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Resultado de Investigación

AN OPTICAL ILLUSION OF VOLUME PROMOTED BY THE TRANSFORMATION OF A PAPER MODEL FROM A SQUARE PRISM TO A TETRAHEDRON

UNA ILUSIÓN ÓPTICA DE VOLUMEN PROMOVIDA POR LA TRANSFORMACIÓN DE UN MODELO DE PAPEL DE UN PRISMA CUADRADO A UN TETRAEDRO

UMA ILUSÃO ÓPTICA DE VOLUME PROMOVIDA PELA TRANSFORMAÇÃO DE UM MODELO DE PAPEL DE UM PRISMA QUADRADO PARA UM TETRAEDRO

Rosa Elena Arroyo-Carmona*, **Jaime Vázquez Bravo****, **Marco A. Mora-Ramírez*****, **Gerardo Paredes-Juárez******, **Aarón Pérez-Benítez*******

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Abstract

During the recycling of a cardboard box in the shape of a right prism with a square base to build a tetrahedron, there was the curious coincidence that the height of the first was equal to the inclined height of the second. After some bending and cutting on the prism, its dynamic conversion to the tetrahedron and vice versa was achieved. This fact motivated us to carry out a study on the meaningful learning of a small group of participants (N = 10) with minimum primary school education, regarding the volume of these two common geometric bodies. It was found that 70 % of the study population thought that the volumes were equal and that 100 % had forgotten the formula to calculate the volume of the tetrahedron. A template to build the model and a video to illustrate the mentioned transformation process are also presented.

Keywords: Meaningful learning. Space geometry. School activities. Teaching model.

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* Doctora en Ciencias. Facultad de Ciencias Químicas, Benemérita Universidad Autónoma de Puebla, México. rosa.arroyo@correo.buap.mx
** Doctor en Ciencias. Instituto Tecnológico Superior de Venustiano Carranza, Puebla, México. drjvazquezb@gmail.com
*** Doctor en Ciencias. Facultad de ciencias químicas, Benemérita Universidad Autónoma de Puebla. marco.morar@correo.buap.mx
**** Maestro en Ciencias. Preparatoria 2 de octubre de 1968. Benemérita Universidad Autónoma de Puebla, México. gerardo.paredesj@correo.buap.mx
*****Maestro en Ciencias. Facultad de Ciencias Químicas. Benemérita Universidad Autónoma de Puebla, México. aaron.perez@correo.buap.mx

Resumen

Durante el reciclaje de una caja de cartón con forma de prisma recto de base cuadrada para construir un tetraedro, se presentó la curiosa coincidencia de que la altura del primero era igual a la altura inclinada del segundo. Luego de algunos dobleces y cortes sobre el prisma, se logró su conversión dinámica al tetraedro, y viceversa. Este hecho motivó la realización de un estudio sobre el aprendizaje significativo, de un pequeño grupo de participantes (N = 10) con escolaridad mínima de primaria, respecto al volumen de estos dos cuerpos geométricos comunes. Se encontró que el 70 % de la población de estudio pensó que los volúmenes eran iguales, y que el 100 % había olvidado la fórmula para calcular el volumen del tetraedro. Además, se presenta una plantilla para construir el modelo y un video para ilustrar el proceso de transformación mencionado.

Palabras clave: aprendizaje significativo, geometría del espacio, actividad escolar, modelo didáctico.

Resumo

Durante a reciclagem de uma caixa de papelão em forma de prisma reto com base quadrada para construir um tetraedro, ocorreu a curiosa coincidência de que a altura da primeira era igual à altura inclinada da segunda. Depois de algumas dobras e cortes no prisma, conseguiu-se sua conversão dinâmica para o tetraedro e vice-versa. Este facto motivou-nos a realizar um estudo sobre a aprendizagem significativo de um pequeno grupo de participantes (N = 10) com escolaridade mínima do ensino básico, relativamente ao volume destes dois corpos geométricos comuns. Constatou-se que 70 % da população do estudo achava que os volumes eram iguais e que 100 % haviam esquecido a fórmula para calcular o volume do tetraedro. Além disso, é apresentado um template para construir o modelo e um vídeo para ilustrar o processo de transformação mencionado.

Palavras chave: Aprendizagem significativo. Geometria sólida. Atividades escolares. Modelo de ensino.

1. Introduction

Geometry of solids is an obligated subject, not only in several branches of science but also in daily life (Booker et al., 2014; Ellenberg, 2021; Morrison, 2000). This is because a characteristic feature of geometry is its dual nature, meaning it has a theoretical domain and practical applications (Fujita & Jones, 2003). For instance, in the packaging and marketing industries, many types of innovative geometric containers are developed to sell commercial products and position their brands. Moreover, they implement the business strategy known as shrinkflation or package downsizing (Durbin & Rourke, 2022; Tisdale, 2023), which consists of modifying the packaging to reduce the quantity of a product, while preserving or slightly increasing the sale price. Sometimes this modification implies changes in the shape of the product's container (Figure 1), so the consumer must be cautious not to be deceived by the visual appearance of the product.

Figure 1. *Two examples of altering the geometry of the containers to hide the decrease in product volume*
Note. Left: Bottles of Gatorade containing 32 *vs* 28 fluid ounces. Modified from Durbin & Rourke (2022); Tisdale (2023).

In the case of chemistry, the understanding of many physicochemical properties of matter depends on the knowledge of molecular geometry; however, although the teaching-learning of geometry is usually addressed in primary education (Beneke, 2016; Huang & Wu, 2019), it has been reported that some students from high school to undergraduate levels did not acquire meaningful learning about the names and three-dimensional shapes of common geometric solids such as the tetrahedron, trigonal bipyramid and octahedron, among others (Arroyo-Carmona et al., 2005; Chamberlin & Candelaria, 2018; Pérez-Benítez et al., 2009; Pinto, 2023).

Based on those results, we have been interested in the teaching-learning of geometry from an early age, by developing low-cost three-dimensional polyhedra models and sharing their construction in science fairs and workshops in primary and secondary schools (Figure 2). We also work with undergraduate chemistry students to identify their misconceptions about spatial geometry. This article presents the construction of a tetrahedron from an empty commercial container, in the shape of a square prism. Furthermore, taking advantage of the visual effect created by the dynamic conversion of one into the other, we analyzed the population's abilities to calculate their volumes.

Figure 2. *The building of three-dimensional models of straws at workshops for children, in primary and secondary schools*

2. The context of work

Good-quality cardboard geometric boxes are often used as packaging for commercial products (such as medicines, gifts, etc.) and then discarded by the user. In an attempt to recycle a square prism (SP) package for building a tetrahedron (T), we found by chance, that the height (h_1) of that SP was equal to the slant height (h_s) of the derived T (Kumar, 2004) (Figures 3.I and 3.II, respectively). The tetrahedron's slant height corresponds to the height of any of its four equilateral triangle faces and must not be confused with the tetrahedron's height, h_2, which is defined as the normal segment that goes from a vertex to the plane formed by the face opposite it (Figure 3.III).

Thus, after eliminating the square bases of SP and graving the lines *a*, on its lateral faces, we performed the dynamic conversion between both polyhedra (See video):

https://youtu.be/kkgTYrVCT3g

At that moment two simple questions emerged: 1) Are the volumes the same for both geometrical bodies? 2) Would common people be capable of determining the volumes for both?

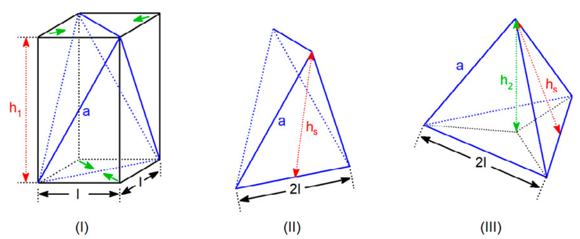

Figure 3. *The transformation of a right square prism (I) into a tetrahedron (II-III) is possible if the diagonal a of I corresponds to edge a of II*
Note. This implies that height h_1 of I equals the slant height, h_s, of II ($h_1 = h_s$). Since the faces of tetrahedron are equilateral triangles, then $a = 2l$. To avoid confusion, the slant and "normal" heights of the tetrahedron are illustrated in III ($h_s \neq h_2$).

3. The geometric properties of a right square prism and tetrahedron

A tetrahedron is a regular polyhedron composed of four equilateral triangles. It has four vertices and three of the four triangles converge at each vertex (Vert, 2022). SP is defined as a three-dimensional geometric shape that has two square bases and four lateral rectangular faces. There are two types of square prisms: Right and oblique prisms (Freitag & Crawford, 2014). The faces in the right square prism, RSP, are orthogonal and it is the one used here. Selected geometric characteristics of RSP and T are outlined in Table 1 (Kumar, 2004).

Table 1. *Selected geometric properties of a right square prism and tetrahedron*

	Right Square Prism (RSP)	Tetrahedron (T)
Edges	12	6
Faces	4 rectangles	4 equilateral triangles
Base(s)	2 squares	Any of 4 equilateral triangles
Height	h_1	h_2=(a/3)(sqrt(6))
Slant height	-	h_s=(a/2)(sqrt(3))
Total area	A=2(l^2) + 4(l)(h_1)	A=(a^2)(sqrt(3))
Volume	V=(l^2)(h_1)	V=(a^3/12)(sqrt(2))

Note. l and h_1 = edge of square face and height of RSP; *a*, h_2 and h_s = edge and "normal" and slant heights of tetrahedron. From Vert (2022); Freitag & Crawford (2014).

3.1 Educational research: Do a right square prism and its derived tetrahedron have equal volume?

The easy conversion of the above-mentioned right square prism into a tetrahedron prompted us to carry out a short study related to the geometrical properties of both polyhedra; specifically, we considered doing a question related to their volumes; however, due to Coronavirus Disease (COVID-19) Pandemic, only ten of our relatives were surveyed.

The inclusion criterion was the participant had attended the 4th grade of primary school onwards.

This study was conducted at home, in an informal session in which the target population was exposed to the dynamic transformation of the model. In this regard, we believe that, given the relative difficulty of the calculation, if the participants had not seen the transformation of the model, they might not have given their answers.

After the participants afforded their alternative ideas to the principal question (2nd column of Table 2), we proceeded to ask them the second question (3rd column of Table 2), being their answers as expected because previously none of them tried to carry out calculations.

Table 2. *Educational background of the study population (N = 10) and their alternative ideas related to the volumes of a right square prism and its derived tetrahedron*

Participant	Educational background of the study population participant	Does the participant believe that both geometrical bodies have the same volume?	The participant remembers the formula of: Tetrahedron (I); square prism (II); both (III); None of them (IV)
A	Primary school	Yes	IV
B	Bachelor	Yes	II
C	Bachelor	No	IV
D	Undergraduate in graphic arts	No	IV
E	Degree in veterinary medicine	Yes	II
F	Degree in public accounting	Yes	IV
G	Degree in Dentistry	Yes	II
H	Degree in dentistry	Yes	IV
I	Ph. D. in medicinal physiology	Yes	II
J	Ph. D. in material science	No	II

4. Results and discussion

It is possible to observe in Figure 4 (top), that only 30% of participants mentioned that the volumes of both geometrical bodies are not equal. Although this answer is correct for question 1, the following answer allowed us to know they just guessed it; this means that they did not remember the formula to calculate the tetrahedron's volume. On the other hand, 50% percent of the population remembered only the way to calculate the square prism's volume (Figure 4, middle) and 100% did not remember the formula to calculate the tetrahedron's volume (Figure 4, bottom).

4.1 Mathematical and physical demonstration of the volume differences between the right square prism and its derived tetrahedron

Once the data were collected, we proceeded to calculate the volume of both polyhedra using the formulas given in the last row of Table 1 (or an online calculator (Casio, 2022)) and the values of $h_1 = 9.35$ cm, $l = 5{,}4$ cm and $a = 10{,}8$ cm. Although the results were mentioned to the study participants, they were not convinced because the tetrahedron's volume (148.46 cm^3) is little more than half of the square prism's volume (272,65 cm^3); this means VT = 54,5 % VRSP.

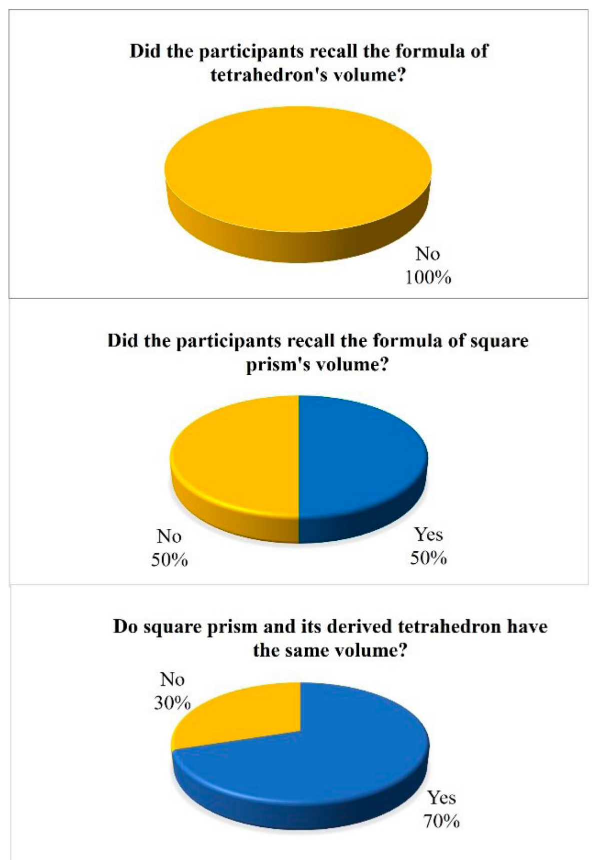

Figure 4. *Top) Alternative ideas about the volume equality of the right square prism and the tetrahedron derived from it; Middle and bottom) The percent of answers corresponding to the knowledge of the formulas indicate that the right square-based prism is easier to remember than a tetrahedron or perhaps that the volume of a tetrahedron was not addressed during their school training*

This fact led us to perform a physical demonstration, such as Archimedes' bathtub historical demonstration of the volume of two irregular objects, and shout "Eureka! Eureka!" (Perkins, 2003); however, that approach is not suitable because our model is made of permeable paper. As an alternative, we make the show by filling the right square prism with rice and then pouring it into the tetrahedron; obviously, a part of the rice was left inside the RSP (Figure 5). Although this experiment is more imprecise than others reported in the literature (Pinilla et al., 2024), we consider it an acceptable demonstration.

Figure 5. *Fraction of the total volume of the square prism (left) after filling its derived tetrahedron with rice (right)*

4.2 Instructions for building the model

1. Look for a recycled right square package and measure the edge of a square face, l.

2. Using the Pythagorean formula, calculate h_1 (See Figure 3 and Equation 1). Remember that *a*=2·l (Kumar, 2004):

$$h_1^2 = l^2 - \frac{l^2}{2^2} = \frac{3}{4}l^2; h_1 = \frac{\sqrt{3}}{2}l = 0.866\left(l\right) \quad (1)$$

3. At height h_1, place a mark and cut out the excess of the package. Also, remove the bottom face.

4. Draw alternately, the diagonals with negative and positive slopes (Figure 6, left). Mark and crease these diagonals in both directions.

Figure 6. *Obtaining a Tetrahedron (left) from a recycled box with the shape of a Right Square Prism (right)*

5. The finished model can be seen in Figure 5 (right) and the transformation of one into the other, in the video.

6. Although part of the spirit of this paper is lost, it means, the recycling, two templates for building RSP and T are provided in Figure 7. Just cut out the templates, fold them by dotted lines and glue them by the corresponding flaps.

7. For the RSP model, it is suggested to glue the lower face while leaving the top unattached. This will allow for both models to be filled, resulting in an objective demonstration.

5. Conclusion

The morphology, usage and recycling of commercial packages help the students to understand the importance of geometry in our daily lives. Additionally, the optical effect produced by the dynamic conversion of a square prism packaging into a tetrahedron raises doubts about the relationship between their areas and volumes; but at the same time, motivates the student to board the mathematical calculations of their volumes.

Moreover, the physical demonstration of their volume differences promotes significant learning and allows us to link a historical (Eureka!) moment

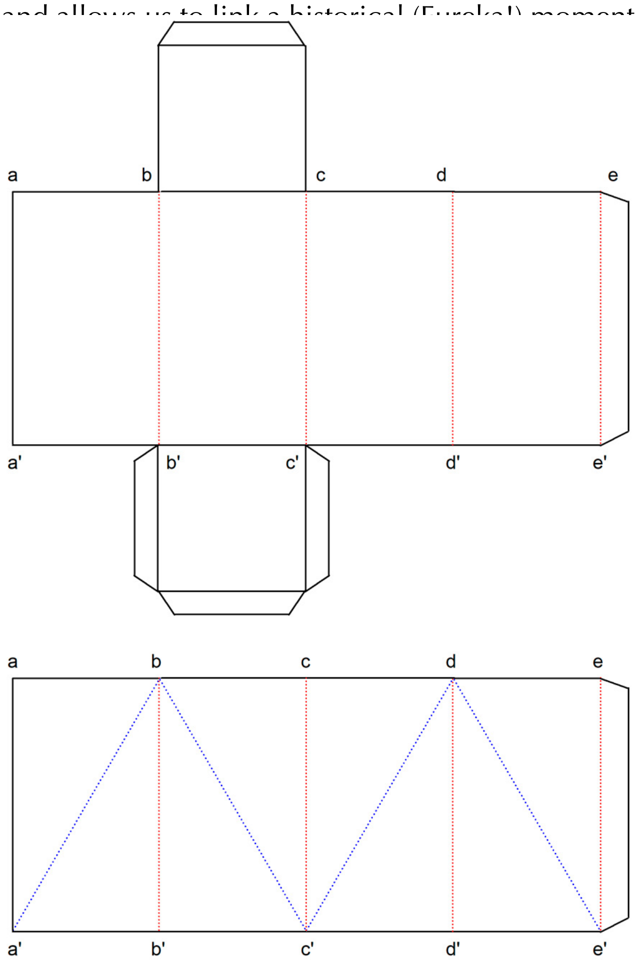

Figure 7. *Templates for building RSP (Top) and T (Bottom) models*

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