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Research Article

T-overlap T-migrative Functions: A Generalization of Migrativity in t-Overlap Functions

Funciones t-migrativas t-overlap: una generalización de migratividad en funciones t-overlap

Funções T-sobrepostas T-sobrepostas: uma generalização da migração para funções T-sobrepostas

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Abstract

This paper introduces a generalization of migrative functions by extending the conditions of the product operation applied in the variables. We operate a number with the variables according to a t-norm instead of multiplying the variable x by this number. Such generalization, whenever it occurs, is called a t-migrative function with respect to such t-norm. Furthermore, we analyse the main properties of t-migrative and t-overlap functions. We introduce some interesting methods of construction of such functions.

Keywords: Migrative function, overlap function, t-norm.

Resumen

Este artículo introduce una generalización de funciones migrativas por extensión de la condición de la operación producto aplicada en las variables. Más específicamente, en lugar de exigir multiplicar la variable x por un número real α, en este trabajo se trabaja este número α con las variables de acuerdo a una t-norma. Se denomina a esta generalización función t-migrativa con respecto a tal t-norma. Luego se analizan las propiedades principales de funciones t-migrativas en funciones t-overlap y se introducen algunos métodos de construcción de este tipo de funciones.

Palabras clave: función migrativa, función overlap, normas triangulares.

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Resumo

Este artigo apresenta uma generalização das funções migratórias estendendo as condições da operação do produto aplicada nas varivéis. Mais especificamente, em vez de exigir a multiplicação da varivél x por um número real α, neste trabalho operamos isso α número com as variáveis de acordo com a norma t. Chamamos essa generalização de uma função t-migrative em relação a essa norma t. Em seguida se analiza as principais propriedades das funções t-sobreposição migratórias e introduzimos alguns métodos de construoção.

Palavras-chaves: função migratória, função de sobreposição, norma t .

Introduction

The purpose of this paper is to generalize the notion of migrative functions by relaxing one of the conditions of product operators applied in the variables. The notion of migrative function was introduced and studied in (Bustince, Montero and Mesiar, 2009). We follow a different approach. Rather than multiplying the variables of the function by a real number a, we operate this number with a t-norm.We allow it some kind of threshold, defined in terms of a t-norm T. We call such generalization a t-migrative function with respect to T and here after, we apply the migrativity to t-overlap functions. We observed that, this simple generalization allows us to state several interesting properties, which can be applied in fuzzy rule-based systemto eliminated bad rules when computing the compatibility degree. Section 1 presents some preliminary concepts. In Section 2, we propose some method of construction of such function. We also study the main properties of the generalized functions.

1. Preliminaries

1.1. Triangular Norms

One of the basic concepts of the fuzzy theory is triangular norms or t-norms. In this paper, these functions are frequently used. We begin with the basic definition of t-norms.

Definition 1 *(See Bustince, Burillo, and Soria (2003)) A triangular norm or t-norm is an aggregation function* $T : [0,1]^2 - [0,1]$ *such that:*

(i) $T(x,1) = x$ *for all* $x \in [0,1]$, $= G(x, 1.y)$

(ii) $T(x,y) \leq T(z,u)$ *if* $x \leq z$ *and* $y \leq u$,

(iii) $T(x,y) = T(y,x)$ *for all* $x, y \in [0,1]$,

(iv) $T(T(x,y),z) = T(x,T(y,z))$ *for all* $x, y, z \in [0,1]$.

We then present some examples of t-norms which are of great interest.

Example 1 *The function* $T : [0,1]^2 - [0,1]$*, defined by* $T(x,y) = \text{mín}\{x,y\}$ *is a t-norm.*

Definition 2 *A t-norm function is strict if it is strictly increasing for its two variables, that is to say if* $x_1 < x_2$ *and* $y \neq 0$ *then* $T(x_1,y) < T(x_2,y)$.

Definition 3 *A t-norm function is positive if* $T(x,y) = 0$ *iff* $xy = 0$.

1.2. Migrativity

The concept of α-migrativity was introduced by Durante and Sarkoci (2008). A bivariate operation's class having a property previously presented by Mesiar and Novak (1996) which was later studied by Fodor and Rudas (2007).

Definition 4 *(Durante and Sarkoci 2008) Let* $\alpha \in [0,1]$ *be fixed. A bivariate operation* $G : [0,1]^2 \to [0,1]$ *is* α*-migrative if* $G(\alpha x, y) = G(x, \alpha y)$, *for all* $x, y \in [0,1]$.

From the definition above, we can easily observed that all function $G : [0,1]^2 - [0,1]$ is 1-migrative, as $G(x,y) = G(1.x,y) = G(x,1.y)$. This definition is referred to as a predetermined α. We generalized the concept of α-migrativity the next form,

Definition 5 *(See Bustince, Montero and Mesiar (2009)) A function* $G : [0,1]^2 - [0,1]$ *is called migrative if and only if* $G(\alpha x, y) = G(x, \alpha y)$ *for all* $x, y \in [0,1]$ *and for all* $\alpha \in [0,1]$.

Example 2 *The function* $h : [0,1]^2 - [0,1]$, *defined by* $h(x,y) = xy$, *is migrative.*

Example 3 *La function* $G : [0,1]^2 - [0,1]$, *defined by* $G(x,y) = \frac{x+y}{2}$, *shows that*

$$\begin{aligned} G(1.x, y) &= \frac{1.x + y}{2} \\ &= \frac{x + 1.y}{2} \end{aligned}$$

thus G is a 1-migrative function, but if you take $\alpha = \frac{1}{2}$, it shows that $G(\frac{1}{2}x, y) \neq G(x, \frac{1}{2}y)$. This is easy to see, in particular if it is done $x = 1$ and $y = \frac{1}{2}$.

The following lemma is the characterization of the main migrative functions.

Lemma 1 *(Bustince, et al., 2010) A function*

$$G : [0, 1]^2 - [0, 1]$$

is migrative if and only if $G(x, y) = G(1, xy)$, for all $x, y \in [0, 1]$.

From the lemma (Bandler and Kohout,1980), we deduce the following corollary, in which a characterization of bivariate migrative function is seen as a function of a variable.

Corollary 1 *(Bustince, et al., 2010) A function $G : [0, 1]^2 - [0, 1]$ is migrative if and only if ther exists a function $g : [0, 1] - [0, 1]$ such that $G(x, y) = g(xy)$ for all $x, y \in [0, 1]$.*

1.3. Overlap and T-Overlap Functions

In this section, we present the definition of overlap function as well as some of its properties. This type of functions constitutes one of the most important pillars in this work.

Definition 6 *(Bustince, et al., 2010) A function $G_S : [0, 1]^2 - [0, 1]$ is an overlap function, if it satisfies the following conditions:*

(G_s1) G_s *is symmetrical,*

(G_s2) $G_s(x, y) = 0$ *if and only if* $xy = 0$,

(G_s3) $G_s(x, y) = 1$ *if and only if* $xy = 1$,

(G_s4) G_s *is not decreasing,*

(G_s5) G_s *is continuous.*

Example 4 *An example of an overlap function is the product function $h(x, y) = xy$, with $x, y \in [0, 1]$.*

Example 5 *The function $G(x, y) = \sin(\frac{\pi}{2}xy)$ is an overlap function.*

Example 6 *Other example of overlap function is $G(x, y) = \tan(\frac{\pi}{4}xy)$.*

The concept of the generalization of the overlap function, is obtained by changing the condition (G_s2) given in (Alcala-Fdez, Alcala and Herrera, 2011). This property requires that given an overlap function G_S, then $G_S(x, y) = 0 \Leftrightarrow xy = 0$. In this generalization, the product operation is replaced by a t-norm $T : [0, 1]^2 - [0, 1]$.

Definition 7 *Let $T : [0, 1]^2 - [0, 1]$ be a t-norm. A function $G_T : [0, 1]^2 - [0, 1]$ is said to be a t-overlap function with respect to T if the following conditions holds:*

(G_T1) $G_T(x, y) = G_T(y, x)$,

(G_T2) $G_T(x, y) = 0 \Leftrightarrow T(x, y) = 0$,

(G_T3) $G_T(x, y) = 1 \Leftrightarrow x = y = 1$,

(G_T4) G_T *is increasing,*

(G_T5) G_T *is continuous.*

2. T-Migrativity

We then present a generalization of the migrativity concept where the multiplication operation is replaced by a t-norm.

Definition 8 *A two-dimensional G function is said to be t-migrative with respect to a t-norm T if for all $\alpha \in [0, 1]$ we have that $G(x, T(\alpha, y)) = G(T(x, \alpha), y)$ for all $x, y \in [0, 1]$.*

Traditionally, a migrativity property is given for the particular case in which $\alpha = 0$.

Proposition 1 *A function $G : [0, 1] - [0, 1]$ is 0-t-migrative if and only if $G(x, 0) = G(0, y)$*

Proof 1 *If G is 0-t-migrative then $G(x, 0) = G(x, T(0, y)) = G(T(x, 0), y) = G(0, y)$ If $G(x, 0) = G(0, y)$ then as for all t-norm $T(0, x) = 0$ then $G(x, T(0, y)) = G(T(x, 0), y)$.*

The following theorem broadly generalizes theorem 1

Theorem 1 *A function $G : [0, 1]^2 - [0, 1]$ is t-migrative with respect to a t-norm T if and only if there exists a function $g : [0, 1] - [0, 1]$ such that $G(x, y) = g(T(x, y))$.*

Proof 2 *Let $G(x, y) = g(T(x, y))$, then*

$$G(x, T(y, z)) = g(T(x, T(y, z))) = g(T(T(x, y), z)) = G(T(x, y), z).$$

If G is t-migrative with respec to a t-norm T, then

$$G(x, y) = G(T(x, 1), y) = G(1, T(x, y))$$

for all $x, y \in [0,1]$, if $G(x,y) = G(u,v)$ then $G(1, T(x,y)) = G(1, T(u,v))$ where $T(x,y) = T(u,v)$ thus g is well defined

The next corollary is a generalization of corollary 1

Corollary 2 *G is t-migrative with respec to a t-norm T if and only if*

$$G(x,y) = G(1, T(x,y))$$

The following corollary shows an obvious consequence of t-migrativity.

Corollary 3 *If G is t-migrative with respect to a t-norm T then G is symmetrical.*

Proof 3 $G(x,y) = G(1, T(x,y)) = G(1, T(y,x)) = G(y,x)$.

Below is a list of some of the properties of t-migrative functions.

Theorem 2 *Let $G : [0,1]^2 - [0,1]$ be a t-migrative function, then:*

1. *G is non-decreasing if and only if g is non-decreasing.*
2. *G is strictly increasing in $[0,1]^2$ if and only if g and T are strictly increasing.*
3. *$G(1,1) = 1$ if and only if $g(1) = 1$.*
4. *$G(0,0)$ if and only if $g(0)$.*
5. *G is continuous if and only if g and T are continuous.*

Proof 4

1. *Suppose that G is a non-decreasing function. Let $x, y \in [0,1]$ such that $x \leq y$, then $G(x,1) \leq G(y,1)$, thus $g(T(x,1)) \leq g(T(y,1))$, therefore $g(x) \leq g(y)$. Suppose that g is a non-decreasing function and $x, y \in [0,1]$ such that $x \leq y$ then for all $z \in [0,1]$ is true that $T(x,z) \leq T(y,z)$, therefore $g(T(x,z)) \leq g(T(y,z))$ thus $G(x,z) \leq G(y,z)$.*
2. *Analogous.*
3. $G(1,1) \Leftrightarrow g(T(1,1)) = 1 \Leftrightarrow g(1) = 1$
4. $G(0,0) \Leftrightarrow g(T(0,0)) = 0 \Leftrightarrow g(0) = 0$
5. *G is continuous if and only if g and T are continuous.*

One of the most important aspects of this work is the generalization of the migratives of overlap functions. One of the results of this generalization is shown.

Theorem 3 *If G_T is a t-overlap function with respect to the continuous t-norm T, then*

$$G(x,y) = G_T(1, T(x,y))$$

is a t-overlap t-migrative function with respect to T.

Proof 5

1. *Evidently G is Symmetrical.*
2. $G(x,y) = 0 \Leftrightarrow G_T(1, T(x,y)) = 0 \Leftrightarrow T(x,y) = 0$.
3. $G(x,y) = 1 \Leftrightarrow G_T(1, T(x,y)) = 1 \Leftrightarrow T(x,y) = 1 \Leftrightarrow x = y = 1$.
4. *G is continuous.*
5. *G is no decreasing.*
   $G(T(x,y), z) = G_T(1, T(T(x,y), z)) = G_T(1, T(x, T(y,z))) = G(x, T(y,z))$.

The following theorem shows that the convex sum of t-overlap t-migrative functions with respect to a continuous t-norm is also t-overlap t-migrative function.

Theorem 4 *If $\alpha_i \geq 0 \forall i = 1, 2, \ldots, n$, $\sum_{i=1}^{n} \alpha_i = 1$, G_i are overlap functions and T is a t-norm continuous, then*

$$G(x,y) = \sum_{i=1}^{n} \alpha_i G_i(1, T(x,y))$$

is t-overlap t-migrative function with respect to T.

Proof 6

1. *G is symmetrical.*
2. *$G(x,y) = 0 \Leftrightarrow \sum_{i=1}^{n} \alpha_i G_i(1, T(x,y)) = 0 \Leftrightarrow \alpha_i G_i(1, T(x,y)) = 0$. Given that $\sum_{i=1}^{n} \alpha_i = 1$ and $\alpha_i \geq 0 \forall i = 1, 2, \ldots, n$ then exist $\alpha_k \neq 0$ thus if $\alpha_k G_k(1, T(x,y)) = 0$ then $G_k(1, T(x,y)) = 0 \Rightarrow T(x,y) = 0$. If $T(x,y) = 0$. then $G_i(1, T(x,y)) = 0$ for all $i = 1, 2, \ldots, n$ thus $G(x,y) = 0$.*
3. *$G(x,y) = 1 \Leftrightarrow \sum_{i=1}^{n} \alpha_i G_i(1, T(x,y)) = 1 \Leftrightarrow \sum_{i=1}^{n} \alpha_i G_i(1, T(x,y)) = \sum_{i=1}^{n} \alpha_i$ thus $\sum_{i=1}^{n} \alpha_i (1 - G_i(1, T(x,y))) = 0$ then $\alpha_i (1 - G_i(1, T(x,y))) = 0$ for all $i = 1, 2, \ldots, n$ since $\alpha_i \geq 0 \forall i = 1, 2, \ldots, n$, and $\sum_{i=1}^{n} \alpha_i = 1$, then exist $\alpha_k \neq 0$ thus if $\alpha_k (1 - G_k(1, T(x,y))) = 0$ then $1 - G_k(1, T(x,y)) = 0 \Rightarrow G_k(1, T(x,y)) = 1 \Rightarrow T(x,y) = 1 \Rightarrow x = y = 1$.*
4. *G is continuous.*
5. *G is non-decreasing.*

Theorem 5 *If T_1 and T_2 are continuous and G_T is a overlap function, then*

$$G(x,y) = G_T(T_1(x,y), T_2(x,y))$$

is a t-overlap t-migrative function with respect to T_1 *or* T_2.

Proof 7 1. G *is symmetrical.*

2. $G(x,y) = 0 \Leftrightarrow G_T(T_1(x,y), T_2(x,y)) = 0 \Leftrightarrow T_1(x,y) = 0 \vee T_2(x,y) = 0$.
3. $G(x,y) = 1 \Leftrightarrow G_T(T_1(x,y), T_2(x,y)) = 1 \Leftrightarrow T_1(x,y) = 1 \wedge T_2(x,y) = 1 \Leftrightarrow x = y = 1$.
4. G *is continuous.*
5. G *is non-decreasing.*

Corollary 4 *If* G *is given as in the previous theorem and* $T_1 = T_2$ *then* G *is t-migrative.*

Proof 8 $G(x, T(y,z)) = G_T(T_1(x, T(y,z)), T_2(x, T(y,z))) = G_T(T_1(T(x,y), z), T_2(T(x,y), z)) = G(T(x,y), z)$.

Corollary 5 *If* T *is a t-norm and a strong negation* n, *then*

$$G(x,y) = \frac{T(x,y)}{T(x,y) + nT(x,y)}$$

is a t-overlap t-migrative function with respect to T.

Theorem 6 *If* G_T *is an overlap function and* T *is a continuous t-norm, then*

$$G(x,y) = 2^{G(1,T(x,y))} - 1$$

is a t-overlap t-migrative function.

Proof 9 1. G *is symmetrical.*

2. $G(x,y) = 0 \Leftrightarrow 2^{G(1,T(x,y))} - 1 = 0 \Leftrightarrow 2^{G(1,T(x,y))} = 1 \Leftrightarrow G(1, T(x,y)) = 0 \Leftrightarrow T(x,y) = 0$.
3. $G(x,y) = 1 \Leftrightarrow 2^{G(1,T(x,y))} - 1 = 1 \Leftrightarrow 2^{G(1,T(x,y))} = 2 \Leftrightarrow G(1, T(x,y)) = 1 \Leftrightarrow T(x,y) = 1 \Leftrightarrow x = y = 1$.
4. G *is non-decreasing.*
5. G *is continuous.*

Theorem 7 *Let* M *be a continuous and increasing fucntion such that* $M(x) = 0 \Leftrightarrow x = 0$ *and* $M(x) = 1 \Leftrightarrow x = 1$. *If* G_T *is an overlap function and* T *is a continuous t-norm, then*

$$G(x,y) = M(G_T(1, T(x,y)))$$

is a t-overlap t-migrative function.

Theorem 8 *Let* M *be a n-dimensional function, non-decreasing, continuous such that* $M(x_1, \ldots, x_n) = 0 \Leftrightarrow x_i = 0$ *for some* $i \in 1, \ldots, n$ *y* $M(x_1, \ldots, x_n) = 1 \Leftrightarrow x_i = 1$ *for some* $i \in 1, \ldots, n$. *Then*

$$G(x,y) = M(G_1, \ldots, G_n)(1, T(x,y))$$

is a t-overlap t-migrative function if G_i *is an overlap function for all* $i \in 1, \ldots, n$ *and continuous t-norm* T.

Proof 10 1. G *is symmetrical*

2. $G(x,y) = 0 \Leftrightarrow M(G_1, \ldots, G_n)(1, T(x,y)) = 0 \Leftrightarrow M(G_1(1, T(x,y)), \ldots, G_n(1, T(x,y))) = 0 \Leftrightarrow \exists k \in 1, \ldots, n$ *such that* $G_k(1, T(x,y)) = 0 \Leftrightarrow T(x,y) = 0$.
3. $G(x,y) = 1 \Leftrightarrow M(G_1, \ldots, G_n)(1, T(x,y)) = 1 \Leftrightarrow M(G_1(1, T(x,y)), \ldots, G_n(1, T(x,y))) = 1 \Leftrightarrow \exists k \in 1, \ldots, n$ *such that* $G_k(1, T(x,y)) = 1 \Leftrightarrow T(x,y) = 1 \Leftrightarrow x = y = 1$.
4. G *is non-decreasing.*
5. G *is continuous.*

Theorem 9 *Let* M *be an n-dimensional and continuous function, such that* $M(x_1, \ldots, x_n) = 0 \Leftrightarrow x_i = 0$ *for some* $i \in \{1, \ldots, n\}$ *y* $M(x_1, \ldots, x_n) = 1 \Leftrightarrow x_i = 1$ *for some* $i \in \{1, \ldots, n\}$. *Then*

$$G(x,y) = M(G_1(1, T_1(x,y)), \ldots, G_n(1, T_n(x,y)))$$

is a t-overlap function with respect to some t-norm T_k *where* T_i *are continuous t-norms and* G_i *are overlap functions.*

Theorem 10 *A function* $G_S : [0,1]^2 - [0,1]$ *is an overlap t-migrative function if and only if* $G_S(x,y) = g(T(x,y))$ *for all* $x, y \in [0,1]$ *holds for some non-decreasing function* $g : [0,1] - [0,1]$ *such that* $g(0) = 0$ *y* $g(1) = 1$.

Proof 11 *Let* G_S *be an overlap t-migrative function, then there exists a non-decreasing function* g *such that* $G_S(x,y) = g(T(x,y))$. *Now* $g(0) = g(T(0,0)) = G_S(0,0) = 0$. *Besides* $g(1) = g(T(1,1)) = G_S(1,1) = 1$ *If* $G_S(x,y) = g(T(x,y))$ *then* $G_S(x, T(y,z)) = G_S(T(x, T(y,z))) = g(T(T(x,y), z)) = G_S(T(x,y), z)$.

Theorem 11 *If* T *is a continuous t-norm and* n *is a strong negation then* $G(x,y) = \frac{T(x,y)}{T(x,y) + nT(x,y)}$ *is a t-overlap t-migrative function with respect to* T.

Proof 12 *By corollary it can be said that* G *is a t-overlap function with respect to the t-norm* T. *On the other hand,* $G(x, T(y,z)) = \frac{T(x,T(y,z))}{T(x,T(y,z)) + nT(x,T(y,z))} = \frac{T(T(x,y),z)}{T(T(x,y),z) + nT(T(x,y),z)} = G(T(T(x,y), z))$.

Theorem 12 *Let $G_1, \ldots, G_n$ be t-migrative overlap functions with respect to T. If $\omega_1, \ldots, \omega_n$ are not negative real numbers such that $\sum_{i=1}^{n} \omega_i = 1$ then*

$$G(x, y) = \sum_{i=1}^{n} \omega_i G_i(x, y)$$

is a t-migrative overlap function with respect the t-norm T.

Proposition 2 *If G_T is a t-overlap function with respect to t-norm T, then G_T is t-migrative with respect to T.*

3. Conclusions

In this article we present a generalization of the concept of migrativity applied to overlap function, this generalization expands and allow the migrativity and the applications of the overlap functions on the topics about artificial intelligence and we present how we can generate new migrative functions.

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