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Artículo de Investigación

Centroid of an Interval Type-2 Fuzzy Set: Continuous vs. Discrete

Centroide de un Conjunto Difuso Tipo-2 de Intervalo: Continuo vs. Discreto

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Abstract

Karnik-Mendel algorithm involves execution of two independent procedures for computing the centroid of an interval type-2 fuzzy set: the first one for computing the left endpoint of the interval centroid (which is denoted by c_l), and the second one for computing its right counterpart (which is denoted by c_r). Convergence of the discrete version of the algorithm to compute the centroid is known, whereas convergence of the continuous version may exhibit some issues. This paper shows that the calculation of c_l and c_r are really the same problem on the discrete version, and also we describe some problems related with the convergence of the centroid on its continuous version.

Key words: Centroid, Karnik-Mendel algorithm, interval type-2 fuzzy set, recursive algorithm.

Resumen

El algoritmo de Karnik-Mendel presenta siempre dos procedimientos independientes para calcular el centroide de un conjunto difuso tipo-2 de intervalo: el primero calculando su extremo izquierdo (denotado como c_l) y el segundo calculando su extremo derecho (denotado como c_r). Esto a´un es cierto en diferentes versiones del algoritmo que han sido propuestas en la literatura. En la versión discreta del centroide no hay problemas relacionados con la convergencia dado que existe un número finito de términos para sumar. Por otro lado, la versión continua tiene algunos problemas relacionados con la convergencia. Este artículo presenta una discusión simple donde se muestra que el cálculo de c_l y c_r en su versión discreta es el mismo problema y no dos problemas diferentes. También se muestran algunos problemas relacionados con la convergencia del centroide en su versión continua.

Palabras clave: Centroide, Algoritmo Karnik-Mendel, Algoritmo recursivo, conjunto difuso tipo-2 de intervalo.

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1. Introduction

The Karnik-Mendel (KM) algorithm was proposed as a method for computing type reduction of interval type-2 (IT2) fuzzy sets [1]. This algorithm has been studied theoretically and experimentally in order to improve its performance on applications. It gives an exact way to get the centroid (if it exists), which is a closed interval, of an IT2 fuzzy set. KM algorithm has two versions: continuous and discrete. The corresponding version is applied on problems depending whether or not the variable's domain is continuous or discrete. Contrary to its discrete counterpart, the continuous version has some problems related to the convergence of the integrals, because they are improper integrals.

Mendel and Liu [2] proved the convergence of the KM algorithm if the centroid exists. An enhanced version is known as *Enhanced Karnik-Mendel* (EKM) algorithm which is 40% faster than KM algorithm [3]. Both versions of this algorithm involve two procedures (even in recent papers [3]): (1) the first one computing c_l, which is the left part of the centroid and, (2) the second one computing c_r, which is the right part of the centroid.

On a discrete domain, Melgarejo *et al.* [4], [5] presented an alternative version of KM algorithm re-expressing the equations for c_l and c_r but still involving two different steps. Separate procedures for computing the centroid of an IT2 fuzzy set have direct implications on engineering applications, such as in [6], where c_l and c_r were calculated by hardware.

According to Mendel and Wu [7] "The computation of L and R represents a bottleneck for interval type-2 fuzzy logic systems", where L and R are two switch points which are found by the KM algorithm. On the other hand, Melgarejo et al. [5] state that "The KM algorithm finds L and R by means of two procedures that *are essentially the same* computationally speaking". The aim of this paper is to show that the preceding sentence is true on the discrete version of the centroid. We will show that the calculation of c_l and c_r is the same problem and therefore that separate procedures are not required to compute the centroid, i.e., equations for c_l and c_r are related and one expression can be deduced from the other one. Also, we will present a simple discussion where the KM algorithm collapses on its continuous version.

2. Continuous version of the centroid

Given an IT2 fuzzy set A (for more details see [8]) which is defined on an universal set $X \subseteq \mathbb{R}$, with membership function $\mu_{\tilde{A}}(x), x \in X$, its centroid (if it exists) $c(\tilde{A})$ is a closed interval $[c_l, c_r]$ in the classical sense of mathematics, i.e.,

$$c(\tilde{A}) = [c_l, c_r]\,,$$

where c_l and c_r are respectively the minimum and maximum of all centroids of the embedded type-1 fuzzy sets in the footprint of uncertainty (FOU) of $\tilde{A}$ (Figure 1(a)). Mendel *et al.* in some papers [2], [9] define continuous version for c_l and c_r of an IT2 fuzzy set $\tilde{A}$:

$$c_l = \min_{l \in \mathbb{R}} \text{centroid}(A_e(l)),$$
$$c_r = \max_{r \in \mathbb{R}} \text{centroid}(A_e(r)),$$

where

$$\text{centroid}(A_e(l)) = \frac{\int_{-\infty}^{+\infty} x\mu_{A_e(l)}(x)dx}{\int_{-\infty}^{+\infty} \mu_{A_e(l)}(x)dx} \tag{3}$$

$$= \frac{\int_{-\infty}^{l} x\overline{\mu}_{\tilde{A}}(x)dx + \int_{l}^{+\infty} x\underline{\mu}_{\tilde{A}}(x)dx}{\int_{-\infty}^{l} \overline{\mu}_{\tilde{A}}(x)dx + \int_{l}^{+\infty} \underline{\mu}_{\tilde{A}}(x)dx}, \tag{4}$$

$$\text{centroid}(A_e(r)) = \frac{\int_{-\infty}^{+\infty} x\mu_{A_e(r)}(x)dx}{\int_{-\infty}^{+\infty} \mu_{A_e(r)}(x)dx} \tag{5}$$

$$= \frac{\int_{-\infty}^{r} x\underline{\mu}_{\tilde{A}}(x)dx + \int_{r}^{+\infty} x\overline{\mu}_{\tilde{A}}(x)dx}{\int_{-\infty}^{r} \underline{\mu}_{\tilde{A}}(x)dx + \int_{r}^{+\infty} \overline{\mu}_{\tilde{A}}(x)dx}, \tag{6}$$

and where $A_e(l)$ and $A_e(r)$ denote embedded type-1 fuzzy sets for which:

$$\mu_{A_e(l)}(x) = \begin{cases} \overline{\mu}_{\tilde{A}}(x), & \text{if } x \leq l, \\ \underline{\mu}_{\tilde{A}}(x), & \text{if } x > l, \end{cases}$$

$$\mu_{A_e(r)}(x) = \begin{cases} \underline{\mu}_{\tilde{A}}(x), & \text{if } x \leq r, \\ \overline{\mu}_{\tilde{A}}(x), & \text{if } x > r. \end{cases}$$

According to Mendel, $l, r \in X$ are switch points, i.e., values of x at which $\mu A_{e(l)}(x)$ and $\mu A_{e(r)}(x)$ switch from $\overline{\mu}_{\tilde{A}}(x)$ to $\underline{\mu}_{\tilde{A}}(x)$ (or vice versa). $\overline{\mu}_{\tilde{A}}(x)$ and $\underline{\mu}_{\tilde{A}}(x)$ are the upper membership function and lower membership function of $\tilde{A}$ (Figure 1(b) and Figure 1(c)).

2.1. Non-existence of the centroid

Mendel *et al.* [2], [9] has studied properties of (4) and (6) assuming existence of the centroid, that is, convergence of the integrals that define it. However, this is not always true and there are some IT2 fuzzy sets for which (4) and (6) do not exist in the sense that they are not finite. One example is the following.

Example 1. Let $\tilde{A}$ be an IT2 fuzzy set (Figure 2) defined over the real numbers $X=\mathbb{R}$ with lower and upper membership functions defined by:

$$\underline{\mu}_{\tilde{A}}(x) = \frac{1}{2}\left(\frac{1}{1+x^2}\right),$$

$$\overline{\mu}_{\tilde{A}}(x) = \frac{1}{1+x^2}.$$

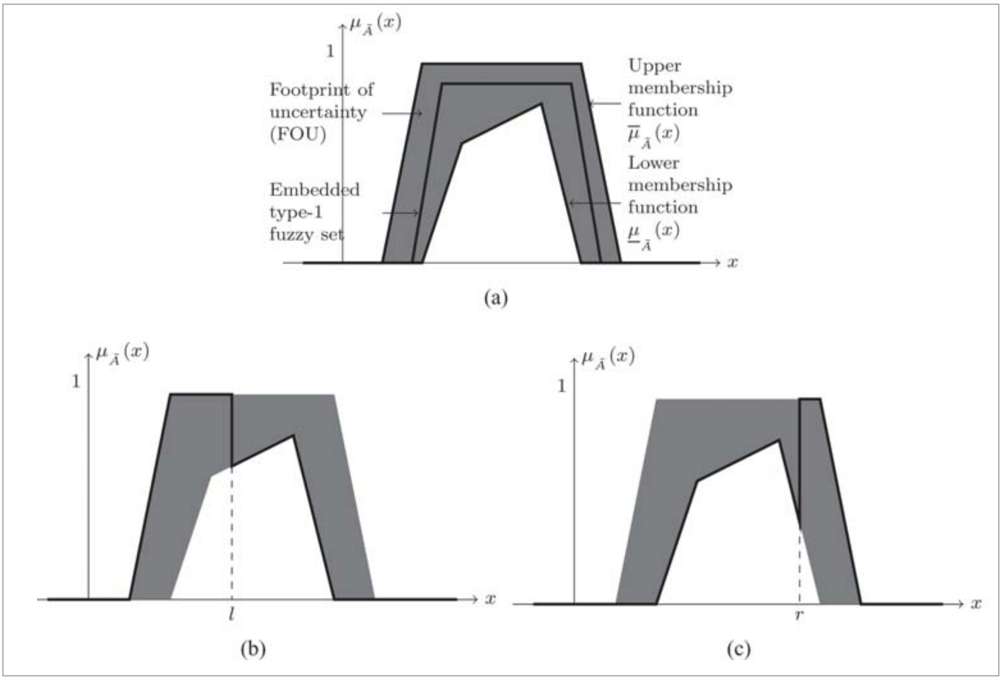

Figure 1. (a) Membership function of an interval type-2 fuzzy set. (b) Interpretation of the switch point l. (c) Interpretation of the switch point r.

Then for a given l:

$$\text{centroid}(A_e(l)) = \lim_{t\to+\infty} \frac{\int_{-t}^{l} x\overline{\mu}_{\tilde{A}}(x)dx + \int_{l}^{t} x\underline{\mu}_{\tilde{A}}(x)dx}{\int_{-t}^{l} \overline{\mu}_{\tilde{A}}(x)dx + \int_{l}^{t} \underline{\mu}_{\tilde{A}}(x)dx}$$

$$= \lim_{t\to+\infty} \frac{\int_{-t}^{l} \frac{x}{1+x^2}dx + \frac{1}{2}\int_{l}^{t} \frac{x}{1+x^2}dx}{\int_{-t}^{l} \frac{1}{1+x^2}dx + \frac{1}{2}\int_{l}^{t} \frac{1}{1+x^2}dx}$$

$$= \lim_{t\to+\infty} \frac{\frac{1}{4}\ln\left(\frac{1+l^2}{1+t^2}\right)}{\frac{1}{2}\arctan(l) + \frac{3}{2}\arctan(t)},$$

but the denominator

$$\frac{1}{2}\arctan(l) + \frac{3}{2}\arctan(t) \to \frac{1}{2}\arctan(l) + \frac{3}{4}\pi$$

and the numerator

$$\frac{1}{4}\ln\left(\frac{1+l^2}{1+t^2}\right) \to -\infty$$

if $t \to +\infty$. Then centroid $(A_e(l)) \to -\infty$ and it does not exist. The calculation of centroid $(A_e(r))$ is similar and although not shown, we claim that centroid $(A_e(r)) \to +\infty$. In this case (1) and (2) do not make sense.

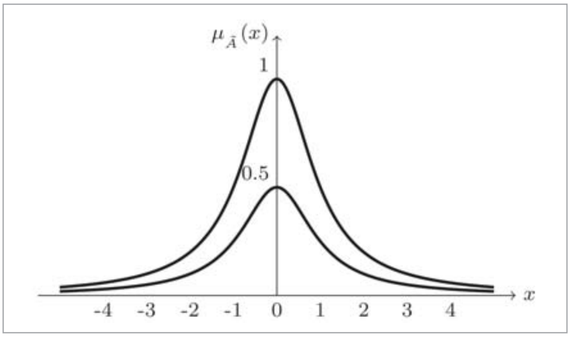

Figure 2. Membership function of an IT2 fuzzy set *A* which does not have centroid. In this case $\overline{\mu}_{\tilde{A}}(x) = 0.5/(1+x_2)$ and $\underline{\mu}_{\tilde{A}}(x) = 1/(1+x_2)$, for all x ∈ $X = \mathbb{R}$

2.2. Continuous version of the KM algorithms

Karnik-Mendel (KM) algorithms for computing c_l and c_r are so similar, that we will refer only to the c_l procedure for the sake of brevity (for more details see [2]):

1. Compute the initial value, c_o, for c_l, as

$$c_0 = \frac{\int_{-\infty}^{+\infty} x \frac{\overline{\mu}_{\tilde{A}}(x) + \underline{\mu}_{\tilde{A}}(x)}{2} dx}{\int_{-\infty}^{+\infty} \frac{\overline{\mu}_{\tilde{A}}(x) + \underline{\mu}_{\tilde{A}}(x)}{2} dx} = \frac{\int_{-\infty}^{+\infty} x(\overline{\mu}_{\tilde{A}}(x) + \underline{\mu}_{\tilde{A}}(x))dx}{\int_{-\infty}^{+\infty} (\overline{\mu}_{\tilde{A}}(x) + \underline{\mu}_{\tilde{A}}(x))dx}$$

and then set $j = 1$ and

$$l_1 = c_0.$$

2. Compute centroid ($A_e(l)$) as

$$\text{centroid}(A_e(l_j)) = \frac{\int_{-\infty}^{l_j} x\overline{\mu}_{\tilde{A}}(x)dx + \int_{l_j}^{+\infty} x\underline{\mu}_{\tilde{A}}(x)dx}{\int_{-\infty}^{l_j} \overline{\mu}_{\tilde{A}}(x)dx + \int_{l_j}^{+\infty} \underline{\mu}_{\tilde{A}}(x)dx}.$$

3. If convergence has occurred, stop. Otherwise, go to step 4.

4. Set

$$l_{j+1} = \text{centroid}(A_e(l_j)).$$

5. Set $j = j + 1$, and go to step 2.

Now we show an example where the preceding algorithm collapses.

Example 2. Let $\tilde{A}$ be the IT2 fuzzy set presented in Section 2.1. One problem arises when we want to find c_l for this fuzzy set. In the first step it is clear that c_o exists and it is given by:

$$
\begin{aligned}
c_0 &= \frac{\int_{-\infty}^{+\infty} x(\overline{\mu}_{\tilde{A}}(x) + \underline{\mu}_{\tilde{A}}(x))dx}{\int_{-\infty}^{+\infty} (\overline{\mu}_{\tilde{A}}(x) + \underline{\mu}_{\tilde{A}}(x))dx} \\
&= \lim_{t\to+\infty} \frac{\frac{3}{2}\int_{-t}^{t} \frac{x}{1+x^2}dx}{\frac{3}{2}\int_{-t}^{t} \frac{1}{1+x^2}dx} \\
&= \lim_{t\to+\infty} \frac{0}{3\arctan(t)} \\
&= 0.
\end{aligned}
$$

So we set $j = 1$ and $l_1 = c_o = 0$. In the second step, as we showed in Section 2.1, centroid $(A_e(l_1))$ = centroid $(A_e(0))$ does not exist (it is not finite). In this case the KM algorithm for c_l collapses. The reader should take note that it does not matter which initial value $l_1 = c_o$ is used (initialization point), in the second step centroid $(A_e(l_1))$ is not finite.

3. Discrete version of the centroid

Karnik and Mendel [1] demonstrated that c_l and c_r can be computed from the lower and upper membership functions of $\tilde{A}$ as follows:

$$c_l = \min_{L\in\mathbb{N}} \text{centroid}(A_e(L)), \tag{7}$$

$$c_r = \max_{R\in\mathbb{N}} \text{centroid}(A_e(R)), \tag{8}$$

were

$$\text{centroid}(A_e(L)) = \frac{\sum_{i=1}^{L} x_i\overline{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^{N} x_i\underline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{L} \overline{\mu}_{\tilde{A}}(x_i) + \sum_{i=L+1}^{N} \underline{\mu}_{\tilde{A}}(x_i)}, \tag{9}$$

$$\text{centroid}(A_e(R)) = \frac{\sum_{i=1}^{R} x_i\underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^{N} x_i\overline{\mu}_{\tilde{A}}(x_i)}{\sum_{i=1}^{R} \underline{\mu}_{\tilde{A}}(x_i) + \sum_{i=R+1}^{N} \overline{\mu}_{\tilde{A}}(x_i)}, \tag{10}$$

and where $L \in \mathbb{N}$ is the switch point that marks the change from $\overline{\mu}_{\tilde{A}}$ to $\underline{\mu}_{\tilde{A}}$ (Figure 3(a)), and $R \in \mathbb{N}$ is the switch point that marks the change from $\overline{\mu}_{\tilde{A}}$ to $\underline{\mu}_{\tilde{A}}$ (Figure 3(b)). $N \in \mathbb{N}$ is the number of discrete points on which the x-domain of $\tilde{A}$ has been discretized. It is assumed that in (9) and (10) $x_1 < x_2 < \ldots < x_N$, in which x_1 denotes the smallest sampled value of x and x_N denotes the largest sampled value of x [3].

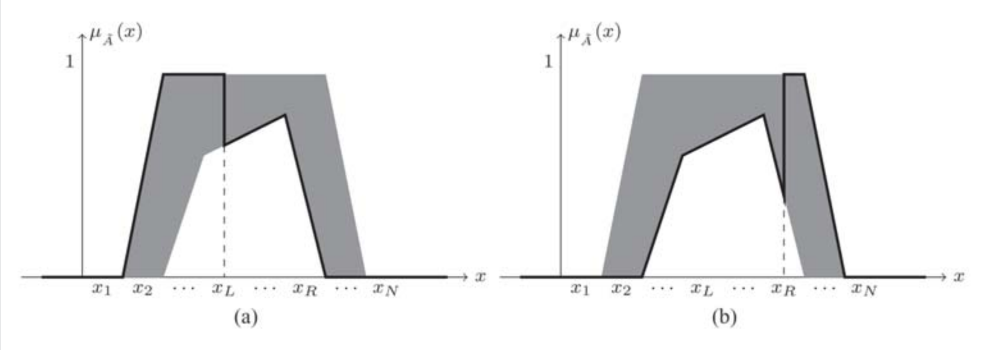

Figure 2. (a) Interpretation of the switch point *L*. (b) Interpretation of the switch point *R*.

3.1. Discrete version of the KM algorithm and recursive algorithm

In order to find *L*, and consequently c_l, the KM algorithm [2] goes as follows:

1. Start the search by computing an initial point c':

$$c' = \sum_{i=1}^{N} x_i\theta_i \Big/ \sum_{i=1}^{N} \theta_i ,$$

with

$$\theta_i = (\underline{\mu}_{\tilde{A}}(x_i) + \overline{\mu}_{\tilde{A}}(x_i))/2, \quad i = 1, 2, \ldots, N.$$

2. Find k $(1 \leq k \leq N - 1)$ such that $x_k \leq c' \leq x_{k+1}$.

3. Set

$$\theta_i = \begin{cases} \overline{\mu}_{\tilde{A}}(x_i), & i \leq k, \\ \underline{\mu}_{\tilde{A}}(x_i), & i > k, \end{cases}$$

and compute

$$c'' = \sum_{i=1}^{N} x_i\theta_i \Big/ \sum_{i=1}^{N} \theta_i$$

4. If $c' = c''$ then stop and set $c_l = c''$, $L = k$. Else go to step 5.

5. Set $c' = c''$ and go to step 2.

The alternative recursive algorithm to compute described in [5] goes as follows:

1. Start by doing:

$$D_0 = \sum_{i=1}^{N} x_i \underline{\mu}_{\tilde{A}}(x_i),$$
$$P_0 = \sum_{i=1}^{N} \underline{\mu}_{\tilde{A}}(x_i),$$
$$c_l = x_{N-1}$$

2. Compute:

$$D_j = D_{j-1} + x_j(\overline{\mu}_{\tilde{A}}(x_j) - \underline{\mu}_{\tilde{A}}(x_j)),$$
$$P_j = P_{j-1} + \overline{\mu}_{\tilde{A}}(x_j) - \underline{\mu}_{\tilde{A}}(x_j),$$
$$c_j = D_j/P_j.$$

3. Check if $c_j < c_l$. If yes, set $c_l = c_j$.

4. Do $j = j + 1$

5. If $j = N - 1$, stop.

3.2. A special property of the discrete version

The following property was first noted in [10]. Let us rewrite the expression (10). If we let $j = N - 1 - i$ then we will have the following:

1. if $1 \leq i \leq R$ then $1 \leq N + 1 - j \leq R$, and hence $N - R + 1 \leq j \leq N$;

2. if $R + 1 \leq i \leq N$ then $R + 1 \leq N + 1 - j \leq N$, and hence $1 \leq j \leq N - R$;

therefore (10) can be written as (by properties of sums)

$$\begin{aligned}
\text{centroid}(A_e(R)) &= \frac{\sum\limits_{j=N-R+1}^{N} y_j \underline{\mu}_{\tilde{A}}(y_j) + \sum\limits_{j=1}^{N-R} y_j \overline{\mu}_{\tilde{A}}(y_j)}{\sum\limits_{j=N-R+1}^{N} \underline{\mu}_{\tilde{A}}(y_j) + \sum\limits_{j=1}^{N-R} \overline{\mu}_{\tilde{A}}(y_j)} \\
&= \frac{\sum\limits_{j=1}^{N-R} y_j \overline{\mu}_{\tilde{A}}(y_j) + \sum\limits_{j=N-R+1}^{N} y_j \underline{\mu}_{\tilde{A}}(y_j)}{\sum\limits_{j=1}^{N-R} \overline{\mu}_{\tilde{A}}(y_j) + \sum\limits_{j=N-R+1}^{N} \underline{\mu}_{\tilde{A}}(y_j)} \\
&= \frac{\sum\limits_{j=1}^{L'} y_j \overline{\mu}_{\tilde{A}}(y_j) + \sum\limits_{j=L'+1}^{N} y_j \underline{\mu}_{\tilde{A}}(y_j)}{\sum\limits_{j=1}^{L'} \overline{\mu}_{\tilde{A}}(y_j) + \sum\limits_{j=L'+1}^{N} \underline{\mu}_{\tilde{A}}(y_j)}
\end{aligned} \tag{11}$$

were

$$y_j = x_{N+1-j}, \quad 1 \leq j \leq N, \tag{12}$$

and

$$L' = N - R\,. \tag{13}$$

Equations (9) and (11) have *the same form*. We can obtain one from the other only with the substitution of x_i by Y_j and L by L' (or vice versa). Equations (9) and (11) differ in and L (switch points) and L' in that the values of x are indexed in reverse order as (12) establishes. Equation (12) means that

$$y_1 = x_N, y_2 = x_{N-1}, \ldots, y_N = x_1,$$

as we show in Figure 4. It is just a permutation (a bijective function) of the N values of x. Equation (12) can be thought as an indexation of the N values of x in reverse order.

It can be seen that the problem for computing c_l and c_r can be reduced to one single procedure. It is just necessary to reverse the order in which the values of x are indexed, and if we are computing c_l then we will need to find a minimum, and if we are computing c_r then we will need to find a maximum. We present a geometrical interpretation in Figure 5(a) and Figure 5(b), where each x_i (y_i) is accompanied by its lower $\underline{\mu}_{\tilde{A}}(x_i)$ ($\underline{\mu}_{\tilde{A}}(y_i)$) or upper $\overline{\mu}_{\tilde{A}}(x_i)$ ($\overline{\mu}_{\tilde{A}}(y_i)$) grade of membership.

If we start form (9) by using a similar argument then we will obtain an analogous expression to (10), i.e., there will be an expression

$$\text{centroid}(A_e(L)) = \frac{\sum_{j=1}^{R'} z_j \underline{\mu}_{\tilde{A}}(z_j) + \sum_{j=R'+1}^{N} z_j \overline{\mu}_{\tilde{A}}(z_j)}{\sum_{j=1}^{R'} \underline{\mu}_{\tilde{A}}(z_j) + \sum_{j=R'+1}^{N} \overline{\mu}_{\tilde{A}}(z_j)} \tag{14}$$

which is analogous to (10), where $z_j = x_{N+1-j}$, $1 \leq j \leq N$, and $R' = N - L$.

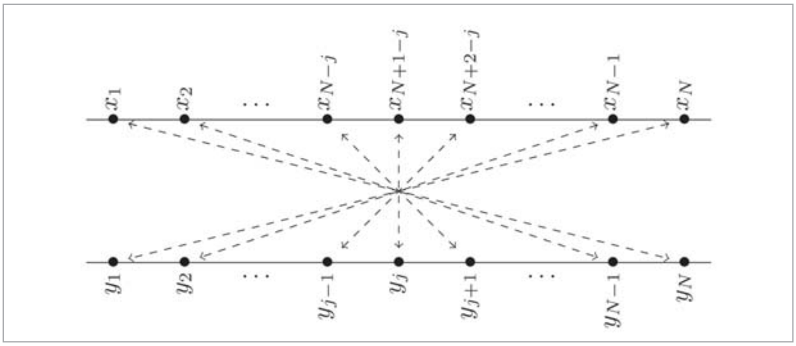

Figure 4. Permutation $y_j = x_{N+1-j}$ $(1 \leq j \leq N)$ that inverts the order in which the values of x are indexed.

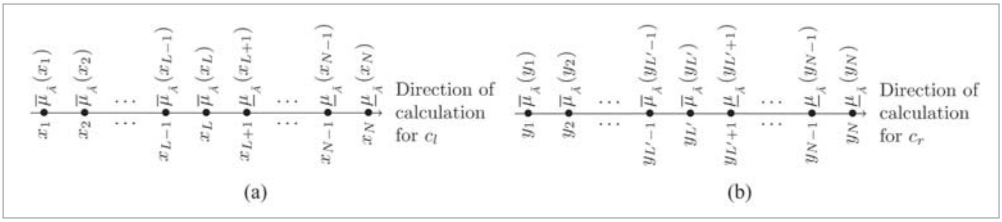

Figure 5. (a) Direction of calculation for computing c_l by using (9).
(b) Direction of calculation for computing c_r by using (11).

3.3. A more general expression

We define a general expression[1] (15) for computing a centroid (c_l or c_r) because of the *duality* between (9) and (11). It is just necessary to replace appropriate values in order to find c_l or c_r as we show in Table I.

$$\text{centroid}(A_e(M)) = \frac{\sum_{i=1}^{M} w_i \overline{\mu}_{\tilde{A}}(w_i) + \sum_{i=M+1}^{N} w_i \underline{\mu}_{\tilde{A}}(w_i)}{\sum_{i=1}^{M} \overline{\mu}_{\tilde{A}}(w_i) + \sum_{i=M+1}^{N} \underline{\mu}_{\tilde{A}}(w_i)} \tag{15}$$

The substitution of M and w_i in (15) by L and x_i respectively gives the expression (9); and the substitution of M and w_i in (15) by $L' (= N - R)$ and $y_i (= x_{N+1-i})$ respectively gives the expression (11) (which is the same Equation (10) as we showed above).

[1] This problem can also be re-formulated with the definition of a general expression by using the duality between(10) and (14).

Table I. Summary for computig a centroid (c_l or c_r) by using (15) based on equations (9) and (11)

c	M	w_i $(1 \leq i \leq N)$	Observation
c_l	L	x_i	If we are finding c_l, we will have to find L such that (15) is minimum by using x_i
c_r	L'	y_i	If we are finding c_r, we will have to find L' $(= N - R)$ such that (15) is maximum by using y_i $(= x_{N+1-i})$

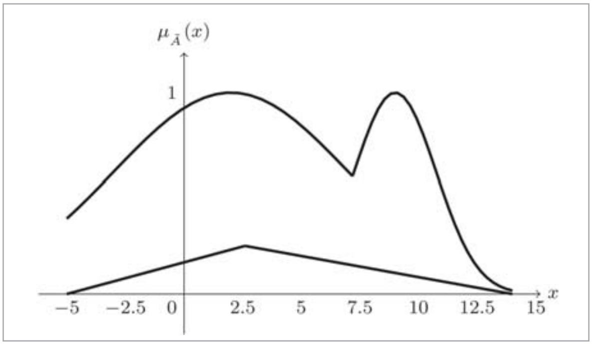

Figure 6. IT2 fuzzy set $\tilde{A}$ with non-symmetric footprint of uncertainty that is defined in the universal set X =[-5, 14].

Example 3. This example is also considered in [2], [4], [5]. Consider an IT2 fuzzy set A with non-symmetric footprint of uncertainty as we show in Figure 6.

The universal set is the closed interval X = [-5, 14]. The lower membership function corresponds to a non-symmetrical triangular membership function

$$\underline{\mu}_{\tilde{A}}(x) = \begin{cases} 0.6(x+5)/19, & \text{if } x \leq 2.6, \\ 0.4(14-x)/19, & \text{if } x > 2.6, \end{cases} \tag{16}$$

whereas the upper membership function is a non-symmetrical Gaussian

$$\overline{\mu}_{\tilde{A}}(x) = \begin{cases} \exp\left(-0.5\left((x-2)/5\right)^2\right), & \text{if } x \leq 7.185, \\ \exp\left(-0.5\left((x-9)/1.75\right)^2\right), & \text{if } x > 7.185. \end{cases} \tag{17}$$

The x-domain of $\tilde{A}$ has been discretized into N = 50 points, then $\Delta x = (14-(-5))/(N-1) = 19/49 = 0.388$. Hence, $x_i = -5+(i-1)\Delta x = -5+(i-1)(19/49)$, where $1 \leq i \leq 50$. Columns 1 and 2 of Table II show all these values. Columns 3 and 4 show $\underline{\mu}_{\tilde{A}}(x_i)$ and $\overline{\mu}_{\tilde{A}}(x_i)$ which have been calculated from (16) and (17). Columns 6, 7 and 8 are the same as columns 2, 3 and 4 respectively, but they were written in reverse order, i.e., row 1 (of columns 6, 7 and 8) corresponds to row 50 (of columns 2, 3 and 4), row 2 corresponds to row 49 and so on. Columns 5 and 9 were calculated with the expression (15). For example, the third value c =3.993 of column 9 was calculated as:

$$a = \overbrace{(14)(0.017) + (13.612)(0.031) + (13.224)(0.054)}^{y_i \overline{\mu}_{\tilde{A}}(y_i)} + \overbrace{(12.837)(0.024) + (12.449)(0.033) + \cdots + (-5)(0.000)}^{y_i \underline{\mu}_{\tilde{A}}(y_i)}$$

$$b = \overbrace{0.017 + 0.031 + 0.054}^{\overline{\mu}_{\tilde{A}}(y_i)} + \overbrace{0.024 + 0.033 + \cdots + 0.000}^{\underline{\mu}_{\tilde{A}}(y_i)}$$

and finally $c = a/b = 3.993$. Similarly, the third value $c = 2.417$ of column 5 was calculated as

$$a = \overbrace{(-5)(0.375) + (-4.612)(0.417) + (-4.224)(0.461)}^{x_i\overline{\mu}_{\tilde{A}}(x_i)} +$$
$$\overbrace{(-3.837)(0.037) + (-3.449)(0.049) + \cdots + (14)(0.000)}^{x_i\underline{\mu}_{\tilde{A}}(x_i)}$$

$$b = \overbrace{0.375 + 0.417 + 0.461}^{\overline{\mu}_{\tilde{A}}(x_i)} + \overbrace{0.037 + 0.049 + \cdots + 0.000}^{\underline{\mu}_{\tilde{A}}(x_i)}$$

and finally $c = a/b = 2.417$.

Melgarejo [5] reports that (with $N = 50$)

1. $c_l = 3.993$ and $c_r = 7.1538$ (KM Algorithms).
2. $c_l = 0.3767$ and $c_r = 7.156$ (Recursive Algorithm).

Table II shows that the minimum value (shaded cell) of column 5 is $c_l = 0.375$ and the maximum value (shaded cell) of column 9 is $c_r = 7.156$, where *both columns* were calculated with the general expression (15).

The reader should take note that this example cannot be solved with the continuous version of the KM algorithm because the integrals cannot be calculated in a closed form.

4. Conclusion

This paper showed that expressions (9) and (10), which were given by Karnik and Mendel in order to calculate c_l and c_r, have the same form with a simple substitution of its index variable. Therefore there is a duality between them and they are not independent. We presented a general dual expression (15) for computing c_l and c_r. It is just necessary to replace appropriate values in order to find c_l or c_r as we showed in Table I.

Finally, we showed that computation of the continuous version of the centroid may exhibit non-existence abnormalities, which do not occur in the discrete version. Simple examples were showed to illustrate the latter issaues.

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Table II. Numerical example (see text for explanation). In this table: $N = 50$, $y_i = x_{51-i}$, $c_l = 0.375$ (shaded cell) and $c_r = 7.156$ (shaded cell)

1	2	3	4	5	6	7	8	9
i	x_i	$\underline{\mu}_{\tilde{A}}(x_i)$	$\overline{\mu}_{\tilde{A}}(x_i)$	c	y_i	$\underline{\mu}_{\tilde{A}}(y_i)$	$\overline{\mu}_{\tilde{A}}(y_i)$	c
1	-5.000	0.000	0.375	3.335	14.000	0.000	0.017	3.896
2	-4.612	0.012	0.417	2.852	13.612	0.008	0.031	3.934
3	-4.224	0.024	0.461	2.417	13.224	0.016	0.054	3.993
4	-3.837	0.037	0.506	2.029	12.837	0.024	0.090	4.090
5	-3.449	0.049	0.552	1.687	12.449	0.033	0.143	4.241
6	-3.061	0.061	0.599	1.390	12.061	0.041	0.217	4.459
7	-2.673	0.073	0.646	1.137	11.673	0.049	0.311	4.747
8	-2.286	0.086	0.693	0.924	11.286	0.057	0.426	5.094
9	-1.898	0.098	0.738	0.751	10.898	0.065	0.555	5.477
10	-1.510	0.110	0.782	0.614	10.510	0.073	0.689	5.862
11	-1.122	0.122	0.823	0.511	10.122	0.082	0.814	6.218
12	-0.735	0.135	0.861	0.439	9.735	0.090	0.916	6.520
13	-0.347	0.147	0.896	0.394	9.347	0.098	0.981	6.758
14	0.041	0.159	0.926	0.375	8.959	0.106	1.000	6.931
15	0.429	0.171	0.952	0.378	8.571	0.114	0.970	7.046
16	0.816	0.184	0.972	0.400	8.184	0.122	0.897	7.114
17	1.204	0.196	0.987	0.439	7.796	0.131	0.789	7.147
18	1.592	0.208	0.997	0.492	7.408	0.139	0.661	7.156
19	1.980	0.220	1.000	0.556	7.020	0.147	0.604	7.152
20	2.367	0.233	0.997	0.630	6.633	0.155	0.651	7.135
21	2.755	0.237	0.989	0.712	6.245	0.163	0.697	7.105
22	3.143	0.229	0.974	0.802	5.857	0.171	0.743	7.061
23	3.531	0.220	0.954	0.897	5.469	0.180	0.786	7.003
24	3.918	0.212	0.929	0.997	5.082	0.188	0.827	6.933
25	4.306	0.204	0.899	1.100	4.694	0.196	0.865	6.851
26	4.694	0.196	0.865	1.204	4.306	0.204	0.899	6.757
27	5.082	0.188	0.827	1.309	3.918	0.212	0.929	6.653
28	5.469	0.180	0.786	1.413	3.531	0.220	0.954	6.540
29	5.857	0.171	0.743	1.515	3.143	0.229	0.974	6.420
30	6.245	0.163	0.697	1.615	2.755	0.237	0.989	6.294
31	6.633	0.155	0.651	1.711	2.367	0.233	0.997	6.161
32	7.020	0.147	0.604	1.803	1.980	0.220	1.000	6.021
33	7.408	0.139	0.661	1.912	1.592	0.208	0.997	5.876
34	7.796	0.131	0.789	2.053	1.204	0.196	0.987	5.728
35	8.184	0.122	0.897	2.220	0.816	0.184	0.972	5.577
36	8.571	0.114	0.970	2.407	0.429	0.171	0.952	5.426
37	8.959	0.106	1.000	2.602	0.041	0.159	0.926	5.274
38	9.347	0.098	0.981	2.794	-0.347	0.147	0.896	5.124
39	9.735	0.090	0.916	2.975	-0.735	0.135	0.861	4.976
40	10.122	0.082	0.814	3.136	-1.122	0.122	0.823	4.831
41	10.510	0.073	0.689	3.273	-1.510	0.110	0.782	4.690
42	10.898	0.065	0.555	3.384	-1.898	0.098	0.738	4.552
43	11.286	0.057	0.426	3.470	-2.286	0.086	0.693	4.420
44	11.673	0.049	0.311	3.533	-2.673	0.073	0.646	4.293
45	12.061	0.041	0.217	3.576	-3.061	0.061	0.599	4.171
46	12.449	0.033	0.143	3.605	-3.449	0.049	0.552	4.055
47	12.837	0.024	0.090	3.622	-3.837	0.037	0.506	3.944
48	13.224	0.016	0.054	3.633	-4.224	0.024	0.461	3.839
49	13.612	0.008	0.031	3.639	-4.612	0.012	0.417	3.739
50	14.000	0.000	0.017	3.645	-5.000	0.000	0.375	3.645

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