VISIÓN ELECTRÓNICA

Exact solutions to helmholtz oscillator equation for an electrical circuit with quadratic nonlinearity

Soluciones exactas a la ecuación del oscilador de helmholtz para circuitos eléctricos con no linealidad cuadrática

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Abstract: In this work we obtain an exact solution to the Helmholtz equation with initial conditions and bounded periodic solutions. This solution is expressed in terms of the Jacobi elliptic function cn. We use this exact solution as a seed to generate a good approximate analytic trigonometric solution to the Helmholtz equation for small values of the modulus. We solve numerically this last equation and we compare this numerical solution with the analytic solution obtained from the solution of the Helmholtz equation. We also give some illustrative examples.

Keywords: Helmholtz equation, Duffing equation, nonlinear circuit, quadratic nonlinearity.

Resumen: En este trabajo se obtiene una solución exacta a la ecuación de Helmholtz con condiciones iniciales y soluciones periódicas acotadas. Esta solución se expresa en términos de la función elíptica de Jacobi cn. Se utiliza esta solución exacta como una semilla para generar una buena aproximación a la solución trigonométrica analítica de la ecuación de Helmholtz para valores de módulo pequeño. Se resuelve numéricamente esta última ecuación y se compara esta solución numérica con la solución analítica obtenida a partir de la solución de la ecuación de Helmholtz. Se dan también algunos ejemplos ilustrativos.

Palabras clave: ecuación de Helmholtz, Ecuación de Duffing, circuito no lineal, no linealidad cuadrática.

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1. Introduction

This paper deals with the Helmholtz oscillator, which is a simple nonlinear oscillator whose equation presents a quadratic nonlinearity and the possibility of escape [1]. In this work our objective is to find, by means of the Jacobi elliptic functions, the exact periodic solution of the nonlinear differential Helmholtz equation with initial conditions that describes the behavior of a nonlinear electrical circuit. Ferroelectric ceramic capacitors are widely used today, in particular in circuits where high capacitance precision is not required. These capacitors have already found also some nontrivial applications as nonlinear elements. They are thus used in electronic starters for fluorescent lamps and in snubbers for power electronic switches. Experimentation with circuits containing such capacitors is also very interesting, [2].

2. Nonlinear electrical circuit

Let us consider a capacitor of two terminals as an dipole in which a functional relationship between the electric charge, the voltage and the time has the following form:

$$f(q,u,t) = 0$$

A nonlinear capacitor is said to be controlled by charge when is possible to express the tension as a function of charge:

$$u = u(q)$$

As an example of nonlinear electrical circuit, let us consider the circuit shown in Figure 1.

This circuit consists on a linear inductor in series with a nonlinear capacitor. The relationship between the charge of the nonlinear capacitor and the voltage drop across it's supposed to have the following quadratic form $u_c = sq + aq^2$

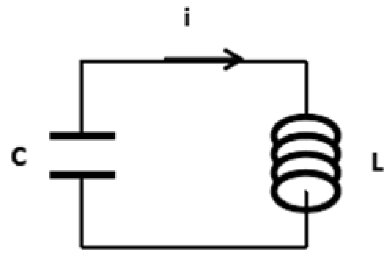

Figure 1. LC circuit.

Here u_c is the potential across the plates of the nonlinear capacitor, q is the charge and s and a are constants.

The equation of the circuit may be written as:

$$L\frac{di}{dt} + sq + aq^2 = 0$$

Where L is the inductance of inductor. Dividing by L and taking into account that $i = \frac{dq}{dt}$ we obtain the Helmholtz oscillator equation

$$\frac{d^2q}{dt^2} + \alpha q + \beta q^2 = 0,$$

Where

$$\alpha = \frac{s}{L}, \beta = \frac{a}{L} \text{ and } a, s = const.$$

3. Exact solutions for the nonlinear model.

We are going to find an exact solution to equation (5) in terms of the Jacobi elliptic function cn. This function is defined as follows:

$$cn(t,m) = cos\phi, where\ t = \int_0^{\phi} \frac{d\theta}{\sqrt{1 - m^2 sin^2\theta}}$$

There are other two important elliptic Jacobi functions: sn and dn. They are defined by

$$sn(t,m) = sin\phi \text{ and } dn\,(t,k) = \sqrt{1 - m^2 sin^2\theta}$$

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The number $m\ (0 < m < 1)$ is called elliptic modulus and the number ϕ is called Jacobi amplitude and it is denoted by am (t, m). Thus,

$$\phi = am(t,m), \sin(\phi) = \sin\big(am(t,m)\big) = sn(t,m)$$

Following identities meet:

$$sn^2(t,m) + cn^2(t,m) = 1, dn^2(t,m) = 1 - m^2 sn^2(t,m)$$

$$\lim_{m\to 0} sn(t,m) = \sin t, \lim_{m\to 0} cn(t,m) = \cos t\,, \lim_{m\to 0} dn(t,m) = 1$$

$$\lim_{m\to 1} sn(t,m) = \tanh t, \lim_{m\to 1} cn(t,m) = \operatorname{sech} t\ \lim_{m\to 1} dn(t,m) = -m^2 sn(t,m)cn(t,m)$$

These functions are derivable and

$$\frac{d}{dt} sn(t,m) = cn(t,m)dn(t,m), \frac{d}{dt} cn(t,m) = -sn(t,m)dn(t,m) = \frac{d}{dt} dn(t,m) = -m^2 sn(t,m)cn(t,m).$$

The graph of functions sn and cn are shown in Figure 2 for $m = \frac{1}{4}$

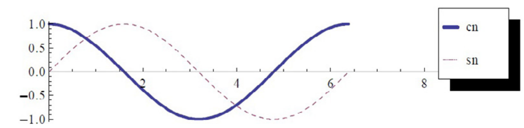


Figure 2. Graph of sn = sn (t, 1/4) and cn = cn(t, 1/4) on the interval0 ≤ t ≤ 8. Source: own.

From Figure 2 we see that functions sn and cn are periodic. They have a common period equal to $4K(1/4) = 4K(m)$, where $K = K(m)$ is called the ellipticK function for modulus m. In our case, $K(1/4) \approx 1.5962422$.

Equations (11) say us that functions sn and cn generalize the sine and cosine functions respectively. In the linear case, the general solution to equation $q''(t) + aq(t) = 0\ is\ q(t) = c_1 \cos(\sqrt{at + c_2})$, where c_1 and c_2 are the constants of integration which are determined from the initial conditions $q(0) = q0$ and $q'(0) = q'0$. When a quadratic term is added we obtain the nonlinear equation

$$q''(t) + aq(t) + \beta q^2(t) = 0$$

and the solution cannot be expressed in terms of cosine function.

Theorem. The solution to the initial value problem

$$q''(t) + \alpha q(t) + \beta q^2(t) = 0, q(0) = q_0 \neq 0, q'(0) = 0$$

Is

$$q(t) = (C + q_0)\varphi^2(t) - C, where\ C = \frac{3\alpha + \sqrt{3}\sqrt{(\alpha - 2\beta q_0)(3\alpha + 2\beta q_0)} + 6\beta q_0}{4\beta}$$

And φ(t) is the solution to the Duffing equation

$$\varphi''(t) + A\varphi + B\varphi^3(t) = 0\ subject\ to\ \varphi(0) = 1\ and\ \varphi'(0) = 0$$

Where

$$A = -\frac{1}{8}(\alpha + 2\beta q_0 - R), B = \frac{1}{12}(3(\alpha + 2\beta q_0) - R), R = \sqrt{3(\alpha - 2\beta q_0)(3\alpha + 2\beta q_0)}$$

$$\varphi(t) = cn\left(\sqrt{A+B} * t, \sqrt{\frac{B}{2(A+B)}}\right) = cn\left(\sqrt{\frac{R + 3(\alpha + 2\beta q_0)}{24}} * t, \sqrt{1 - \frac{2R}{R + 3(\alpha + 2\beta q_0)}}\right)$$

$$q(t) = -\frac{3\alpha + 2\beta q_0 - R}{4\beta} + \frac{3\alpha + 6\beta q_0 - R}{4\beta} * cn^2\left(\sqrt{\frac{R + 3\alpha + 6\beta q_0}{24}} * t, \sqrt{1 - \frac{2R}{R + 3\alpha + 6\beta q_0}}\right)$$

$$R = \sqrt{3(\alpha - 2\beta q_0)(3\alpha + 2\beta q_0}.$$

For small values of $m = \sqrt{1 - \frac{2R}{R + 3\alpha + 6\beta q_0}}$ following is a good trigonometric approximation to the solution to i.v.p (13):

$$trigo(t) = -\frac{3\alpha + 2\beta q_0 - R}{4\beta} + \frac{3\alpha + 6\beta q_0 - R}{16\beta} * [4\cos(wt) - m^2 \sin(wt)(\sin(2wt) - 2wt)]\cos(wt)$$

Where

$$m = \sqrt{1 - \frac{2R}{R + 3\alpha + 6\beta q_0}}, w = \sqrt{\frac{R + 3\alpha + 6\beta q_0}{24}}\ and\ R = \pm\sqrt{3(\alpha - 2\beta q_0)(3\alpha + 2\beta q_0}$$

Let us consider some illustrative examples.

Example 1. Let $q0 = 1, \alpha = -5\ \beta = 7$

The solution to equation $q^{n}(t) - 5(t) + 7q^{2}(t) = 0\ is$

$$q(t) = 0.305351 + 0.694649 cn^{2}(1.19982\, t, 0.56296).$$

$$trigo(t) = 0.305351 + 0.173662\, cos\,(1.19982t)\ (4.cos(1.19982t)\ 0.56296\ sin(1.19982t)\ (sin(2.39965t)\ - 2.39965t))$$

This solution is periodic with period $T = 3.1862$. Its graph is shown in Figure 3.

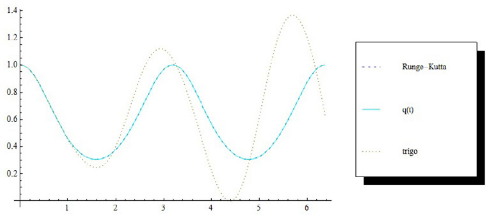


Figure 3. Graph of $q(t) = 0.3 + 0.7\ cn^{2}(1.2t, 0.56))$. Source: own

Example 2. Let $q0 = 1, \alpha = 4\ and\ \beta = 1$

The solution to equation $q^{n}(t) - 4q(t) + q^{2}(t) = 0\ is$

$$q(t) = -1.2087 + 2.20871\ cn^{2}(1.0639\, t, 0.325227).$$

$$trigo(t) = -1.20871 + 0.552178\ cos\,(1.0639t)\ (4.cos(1.0639t) - 0.325227\ sin(1.0639t)\ (sin(2.1278t) - 2.1278t)).$$

This solution is periodic with period $T = 3.25021$. Its graph is shown in Figure 4.

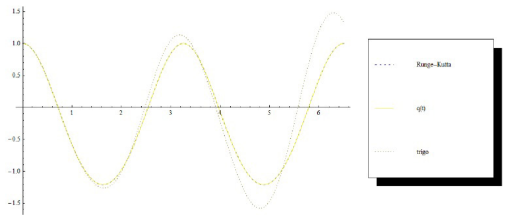


Figure 4. Graph of $q(t) = -1.21 + 2.21cn^{2}(1.06\, t, 0.33)$. Source: own.

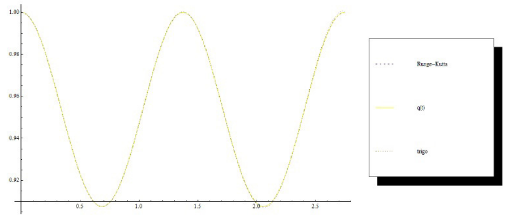
Example 3. Let $q0 = 1, \alpha = -21\ and\ \beta = 22$

The solution to equation $q^{n}(t) - 21q(t) + 22q^{2}(t) = 0\ is$

$$q(t) = 0.90759 + 0.092401cn^{2}(2.3262\, t, 0.0626116).$$

$$trigo(t) = 0.907599 + 0.0231003\ cos(2.3262t)\ (4.cos(2.3262t) - 0.0626116\ sin(2.3262t)\ (sin(4.6524t) - 4.6524t))$$

This solution is periodic with period $T = 1.37245$. Its graph is shown in Figure 5.



. Source: own.

4. Conclusions

We showed that the solution to Helmholtz equation $q''(t) + \alpha q(t) + \beta q^{2}(t) = 0$ may be obtained in the form $q(t) = a\,\phi^{2}(t) + b$, where a and b are some constants and $\phi(t)$ is the solution to some Duffing equation $\phi''(t) + A\phi + B\phi^{3}(t) = 0$ whose solution was studied in previous works [3]-[4]. We may say that the trigonometric solution is very good when the modulus m is relatively small in magnitude. This solution is of great practical application since trigonometric functions are easier to understand and evaluate than the Jacobi elliptic ones. For a more detailed exposition about nonlinear oscillations see, [5].

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