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A CONTEXT VISION

Relations between history of mathematics and training of engineers

Relaciones entre historia de las matemáticas y formación de ingenieros

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ABSTRACT

Integrating historical and epistemological aspects of Newton and Leibniz's mathematics to what is generally known as Fundamental Theorem of Calculus (FTC), allows to improve accessibility to engineering students (understanding it in the college context as "a tool for the preparation of professionals" [1]) to this mathematical object. Indeed, the way Newton and Leibniz found this theorem was a result that emerged when they were solving a problem [2]–[4] this is in conformity with the need of engineering to design artifacts that work in practice, fulfilling the purpose and specifications that motivated it [5]. This writing seeks to collaborate with the transition from Newton and Leibniz's scientific knowledge to current pedagogical knowledge, proposing a new entry to this theorem, via the GeoGebra software. Three relations of the FTC will be presented: its relation with current and previous university textbooks; the FTC in the works of Newton and Leibniz; and, finally, the history and teaching practices of engineering trainers.

RESUMEN

Integrar aspectos históricos y epistemológicos de las matemáticas de Newton y de Leibniz en lo que actualmente conocemos como Teorema Fundamental del Cálculo (TFC), permite mejorar la accesibilidad (entendiéndola como "una herramienta para la preparación de profesionales" [1]) de los estudiantes de ingeniería a este objeto matemático, por cuanto Newton y Leibniz se relacionaron con este teorema como un resultado que emergió al resolver un problema [2]–[4], en concordancia con la necesidad de la ingeniería de diseñar artefactos que funcionen en la práctica, cumpliendo con el propósito y especificaciones que lo motivaron [5]. Este escrito busca entonces colaborar con el tránsito del conocimiento científico de Newton y Leibniz al saber pedagógico actual, proponiendo una nueva entrada a dicho teorema, vía el software GeoGebra. Se presentarán tres relaciones del TC: su relación con los textos universitarios actuales y precedentes; el TFC en los trabajos de Newton y de Leibniz; y, por último, la historia y las prácticas docentes de los formadores de ingenieros.

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1. Introduction

Many people often have two positions about mathematics that harm its teaching and learning process: considering them too difficult, or seeing them only from the utility of its applications [6]. In particular, the FTC does not escape from these negative aspects. Certainly, the FTC is considered complex due to its implicit and explicit mathematical objects (e.g, function, continuity, accumulation, and even the inconvenience of visualizing an area with an integral, and tangent line or rate of change with derivative) or in the best of the cases, because it's only remembered due to the usefulness of the evaluative part [4], [7] - [14].

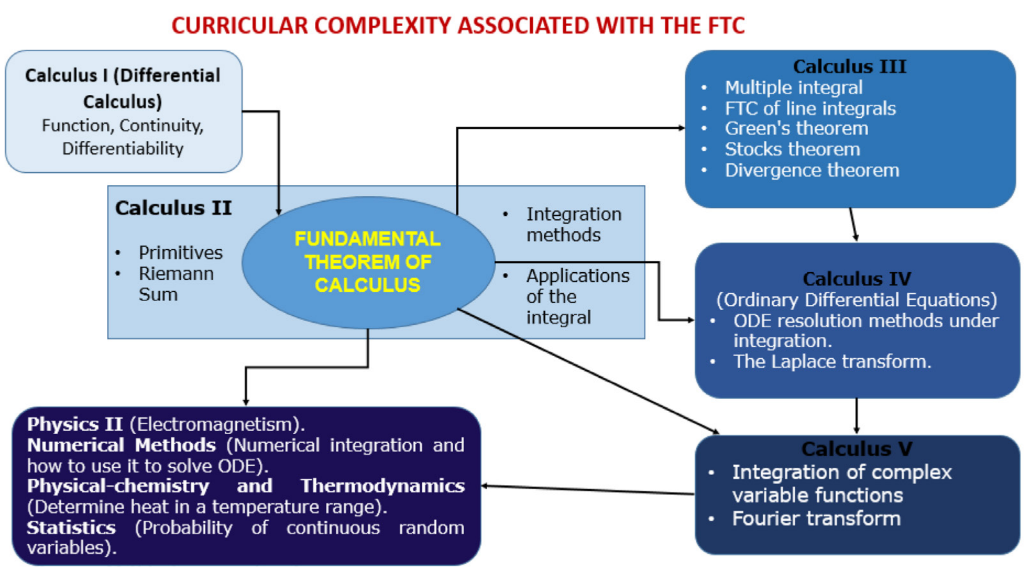
A plausible solution to this problem would be to approach mathematics by its historical development, in order to allow students to see the context and the problem in a more natural way [4], [6], [15]. In addition, it would facilitate understanding the status they have of cultural activity inseparable from other human activities and practices [16]. Therefore, it emerges like a reasonable alternative to tackle work developed by Newton and Leibniz on the FTC.

However, why focus on the points of view of these two mathematicians? Because Newton and Leibniz found the FTC solving a common problem and of great difficulty of the time: the inverse problem of tangents that looked like a generalization of that of finding areas [3], [17], and also, as an aspect of total relevance: both offered what would be recognized today as tables of integrals, thus showing an algorithm for the use of the solution proposed by them [2], [3], [18]. Nevertheless, other famous mathematicians such as Isaac Barrow, Bonaventura Cavalieri, Pierre de Fermat, John Wallis, Hendrick van Heuraet, James Gregory, Leonhard Euler, Joseph-Louis Lagrange, Augustin-Louis Cauchy, etc. cannot be overlooked. They worked with explicit or implicit mathematical objects of the FTC or gave preliminary versions of that theorem [4], [12], [15], [19] - [21].

Another common problem with learning FTC is that engineering students don't understand it like a theorem, nor even like *fundamental* [22]. However, for engineering curricula, the FTC is a main knot in the basic conceptual (or thematic) woven of the different subjects. Figure 1 shows some examples of this type of woven and the importance of FTC in the curricula who are used for the training of engineers.

Three relations of the FTC will be presented in this writing: its relation with current and previous university textbooks; the FTC in the works of Newton and Leibniz; and, finally, the history and teaching practices of engineering trainers. For Newton's and Leibniz's writings, Newton's versions of the FTC will be shown from his manuscript "The October 1666 Tract on Fluxions" [3], [23], [24], and for Leibniz's version the paper of the year 1693 published in "Acta Eruditorum" [4], [12], [21], [25].

Figure 1. Some concepts, theorems, notions, etc. involved with the FTC.



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2. First relation: Current Status.

In this first relation, it will be shown the usual presentations of the FTC in calculus textbooks currently used in engineering curricula. The first question that immediately arises is: which FTC are you talking about? Below is the most complete version of this theorem in James Stewart's book:

Fundamental Theorem of Calculus:

Let be a continuous function on the interval (a,b):

1) If $g(x)=\int_a^x f(t)\,dt$, then $g'(x)=f(x)$.

2) $\int_a^b f(x)dx=F(b)-F(a)$ where F is an antiderivate of f, i,e,, $F'(x)=f(x)$, [26]

However, there is a problem that should be pointed out: each university textbook addresses the FTC from different points of view. For example, they use to alternate the order of 1) and 2), as will be seen in Figure 2. Other way is to vary their presentation, for example, introducing previously the definite integral or the area problem, and later, the integration by substitution or the Riemann sum [27] - [30].

Figure 2. Different Spanish versions of the first part of the FTC in the Zill and Thomas's textbooks, respectively [29], [30].

Theorem 5.5.1 Fundamental Theorem of Calculus—Antiderivative Form

If f is continuous on an interval $[a, b]$ and F is any antiderivative of f on the interval, then

$$\int_a^b f(x)\,dx = F(b) - F(a). \qquad (1)$$

TEOREMA 4 **Teorema fundamental del cálculo, parte 1**

Si f es continua en $[a, b]$, entonces $F(x) = \int_a^x f(t)\,dt$ es continua en $[a, b]$ y diferenciable en (a, b), y su derivada es $f(x)$;

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \qquad (2)$$

Some questions arise from these different points of view: What are the didactical consequences by choosing one of these university textbooks? What happen with the student learning process of the FTC, for curriculum and for teachers? What skills improve or lose the future engineer if one book or another is chosen? What happened if the FTC is presented with one version or another one, or in a different order?

In the context of Integral Calculus, these ways of introducing the FTC in university textbooks [26], [28] - [30] could be called a-chronological, because they do not consider the historical development of implicit and explicit mathematical objects of this theorem and its previous historical versions. In Figure 3, Hairer and Wanner present the chronological sequence that historical development follows in mathematical analysis and the contrast with the line followed by textbooks.

Figure 3. Contrast between the historical development of mathematical analysis topics and their presentation made in university courses [31]

Traditionally, a rigorous first course in Analysis progresses (more or less) in the following order:

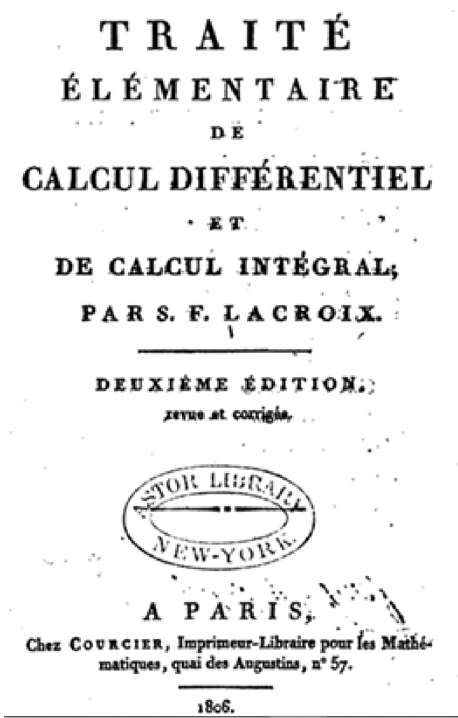
$$\begin{array}{l} \text{sets,} \\ \text{mappings} \end{array} \Rightarrow \begin{array}{l} \text{limits,} \\ \text{continuous} \\ \text{functions} \end{array} \Rightarrow \text{derivatives} \Rightarrow \text{integration.}$$

On the other hand, the historical development of these subjects occurred in reverse order:

$$\begin{array}{l} \text{Cantor 1875} \\ \text{Dedekind} \end{array} \Leftarrow \begin{array}{l} \text{Cauchy 1821} \\ \text{Weierstrass} \end{array} \Leftarrow \begin{array}{l} \text{Newton 1665} \\ \text{Leibniz 1675} \end{array} \Leftarrow \begin{array}{l} \text{Archimedes} \\ \text{Kepler 1615} \\ \text{Fermat 1638} \end{array}$$

However, the presentations of the college textbooks used in the training of engineers in Colombia are relatively new. Indeed, there are historical records with suggestions of the implicit presence of the FTC in the university syllabuses of the XIX century. In the program of Professor Andrés Arroyo in 1887 for Differential and Integral Calculus, specifically in the topics of the Geometric Significance of the Integral and the Approximate Calculus of a Definite Integral, the FTC appeared in an implicit way too, and in addition, he proposed to teach the integration methods (by parts, substitution, partial fractions, etc.) [32]. Explicitly this theorem is found as a Fundamental Theorem in the section called Planimeter Theory in the *Mathematical Analysis* courses at the time [32]. A text that was used at that time and until the middle of the 20th century was the *Cours d'analyse de l'École Polytechnique* by Charles-François Sturm [32]. The next Figure 4 shows how the FTC appeared implicitly in university textbooks at the beginning of the XIX century, relating it more to the problem of squaring (these problems of squaring are geometrical problems that basically consist in making a square to have the same area to that of a given initial figure):

Figure 4. Presentation of the FTC in the Lacroix text on page 321 [33].



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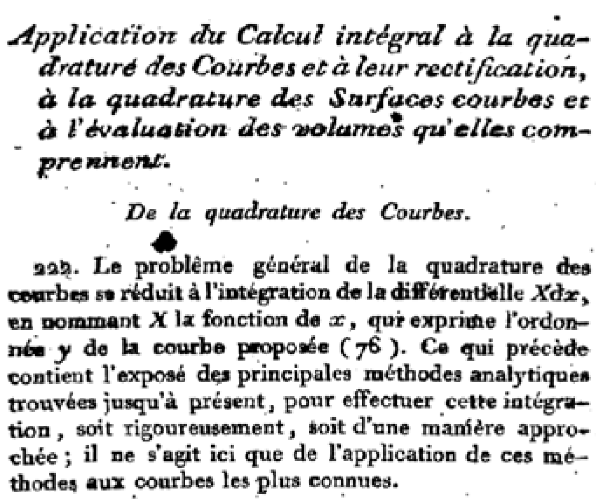


Application du Calcul intégral à la quadrature des Courbes et à leur rectification, à la quadrature des Surfaces courbes et à l'évaluation des volumes qu'elles comprennent.

De la quadrature des Courbes.

222. Le problème général de la quadrature des courbes se réduit à l'intégration de la différentielle Xdx, en nommant X la fonction de x, qui exprime l'ordonnée y de la courbe proposée (76). Ce qui précède contient l'exposé des principales méthodes analytiques trouvées jusqu'à présent, pour effectuer cette intégration, soit rigoureusement, soit d'une manière approchée ; il ne s'agit ici que de l'application de ces méthodes aux courbes les plus connues.

In the second half of the eighteenth century and the beginning of the nineteenth, the influence of Benito Bails's texts on teachers (of mathematical analysis such as José Celestino Mutis) is highlighted by Professor Luis Carlos Arboleda: Mutis favored the "transformations in the local thought of the analytical method by using the mathematical treatises such as Bails's Elements of Mathematics, with an epistemological, cognitive and pedagogical approach different from a traditional course such as Wolff's" [34]. In Bails's textbooks, the FTC don't appear explicit as a theorem, but one can find the integration methods and the calculation of plane areas [35]. As a curious historical fact, it should be clarified that the first one to use this denomination of FTC was Paul du Bois-Reymond in 1876 in an article on Fourier series where he wrote *Fundamentalsatz der Integralrechnung* [4].

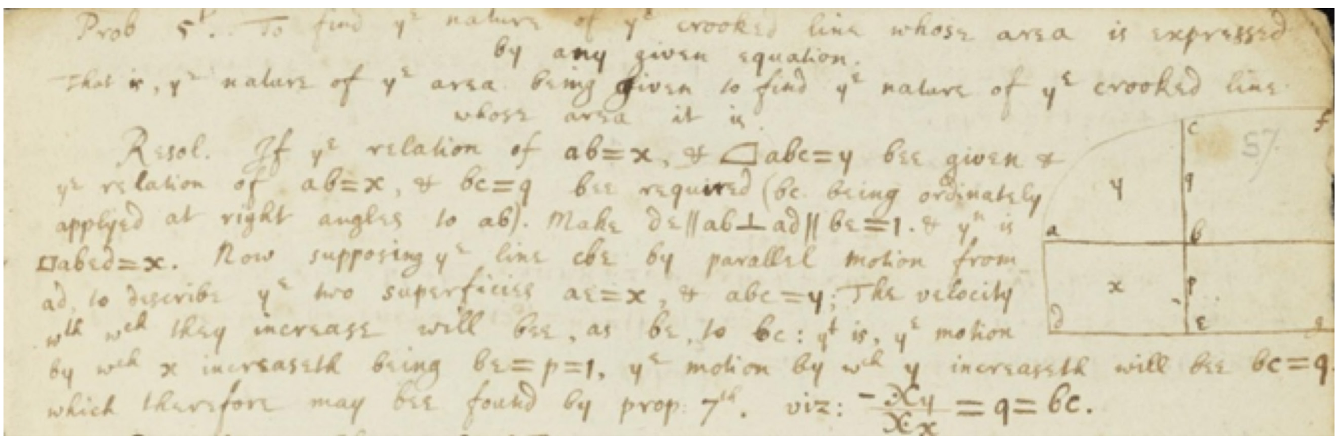
3. Second relation: Historical point of view of the Newton and Leibniz's FTC

3.1. Newton and the FTC

For Newton the FTC emerged when the inverse algorithm that solves the tangents problem solved the areas problem [3], [4], [23], [36]. This historical presentation of the FTC can be found in the manuscript "The October 1666 Tract on Fluxions". However, Newton already had this idea in a general way since 1665 [3], [23]. Later, this inverse relationship between the problem of areas and the problem of tangents is again announced in the book "De Analysi" of 1669 [23].

In Newton's calculus, the motion of a point generates a line, and the motion of a line generates a surface. For him, the quantities generated by a flow are called fluent and their instantaneous velocities are the fluxions. In the manuscript "The October 1666 Tract on Fluxions" he introduced the problem of fluxions and a possible solution in Proposition 7. Then, in Proposition 8 he performed the inverse problem of velocities. The first part of the FTC ensures that the rate of change of the area under a curve is given by the ordinate of the curve that delimits it. In addition, Newton calculated in Problem 5 of his manuscript the derivative of an area under a curve. Don't forget that the algorithmic processes, of what we know today as integral calculus, were already given in Proposition 8.

Figure 5. Problem 5 of the manuscript "The October 1666 Tract on Fluxions", [37].


Prob 5t. To find ye nature of ye crooked line whose area is expressed by any given equation.
That is, ye nature of ye area being given to find ye nature of ye crooked line whose area it is.
Resol. If ye relation of ab=x, & ◁abc=y bee given & ye relation of ab=x, & bc=q bee required (bc being ordinately applyed at right angles to ab). Make de∥ab⊥ad∥be=1. & yn is □abed=x. Now supposing ye line cbe by parallel motion from ad to describe ye two superficies ae=x, & abc=y; The velocity wth wch they increase will bee, as be, to bc: yt is, ye motion by wch x increaseth being be=p=1, ye motion by wch y increaseth will bee bc=q. which therefore may bee found by prop: 7th. viz: $-\frac{\mathfrak{X}y}{\mathfrak{X}x} = q = bc$.

The detailed explanation of Problem 5 made by Newton in 1666 and the analysis of the first example that he offers, can be read in [38]. One of the direct conclusions of this manuscript regarding FTC is that:

$$\frac{d}{dx}\int_a^x f(t)dt=f(x)$$

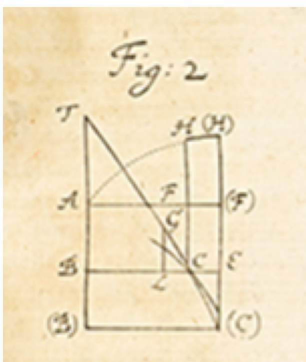
That is, what is currently recognized as the dynamic part of the FTC is shown since Newton found the instantaneous velocity of the area under the *abc* curve.

3.2. Leibniz and the FTC:

Gottfried Leibniz's calculus began with the idea that sum and difference are inverse operations [37]. What we recognize as FTC for Leibniz arises when he managed to find the area under a curve by constructing an auxiliary curve whose slope is proportional to the height of the original curve [4]. The version of the FTC that will be worked on below is from 1693 and appears in "Acta Eruditorum" [25]. However, Leibniz had been developing this idea for a while. Indeed, the first glimpse of his work in this regard is in his writing "Methodus tangentium invers" of 1673, where phrases such as "regress from the tangents or other functions to the ordinates" appear, [39]. Why not present here a version prior to 1693? Because the method used by Leibniz at that time used the application of power series. We'll chosen the geometric input in order to favor visualization, for example, using some mathematical software.

The Figure 6 contains figure 2 of the "Acta Eruditorum", [40]. In this one, the points in parentheses *(H), (F), (C)* and *(B)* are infinitesimal. The curve *AH(H)* is the figure whose area should be found. The curve *C(C)* is the curve whose derivative in *C* is precisely *FH*. It should be noted that Leibniz reached his result by comparing the triangles *TBC* (characteristic triangle) and *CE(C)* (the differential triangle). The whole reconstruction of Leibniz's geometric proof can be seen in [38].

Figure 6. Diagram used by Leibniz for the FTC in the journal Acta Eruditorum 1693 [41].



(Mathematical Treasure: Leibniz's Papers on Calculus)

The direct conclusion of this Leibniz's paper is that he found the second part of the FTC. Indeed, the evaluative part emerges in this way: if you want to find the area under a curve with *y*-ordinate, what you need is to find a curve *z* such what $z=\int ydx$, i.e., if $z=\int ydx$ it has to be found $z=F(x)$ (the primitive) that satisfies:

$$\frac{dz}{dx} = \frac{y}{1} \rightarrow dz = ydx$$

In a particular way, $\int_a^x dz = \int_a^x ydx$, or analogically:

$$F(x) = \int_a^x dz = \int_a^x ydx = \int_a^x f(x)dx.$$

In that way, $F(a)=0$ then $F(x)-F(a)=\int_a^x f(x)\,dx$.

4. Third relation: The history and teaching practices of engineering trainers (use of technological tools-GeoGebra).

First, it is clear that most teachers make use of technological tools in education as a support instrument to guide their courses, and for the construction and elaboration of didactical material and to search information in websites [42]. However, it is recommended that the experiences with that technological tool be longer, with a good impact on the content, in order to have greater articulation and integration with the curriculum [43]. So, how to choose an ideal software for a teaching practice?

It must be pay attention how complex this task is, because developments in educational technology and the creation of many software cause variations in the teaching of mathematics by engineering students, which yields the construction of new techniques and methods to solve or address certain problems [44]. Furthermore, the selection of educational software is NOT easy. One way to tackle this problem is considering other aspects, questions and criteria suggested by educational research (focused on this field) on the benefits and attributes of each technological program:

- The works developed in this software look like attractive? (Attractive in the sense that it uses graphics, movements, videos or audios).

- Does this software allow you to move freely over the graphics or videos made?

- Does the software produce valid results?

- Is this software versatile, easy to operate, robust (which repels the appearance of errors while it is running and avoids frequent system updates)? [45]

In Colombia, the most of population of university students belonging to the middle and lower socio-economic status, so it should be taken into account that the chosen software must be freely distributed too.

Another way that is suggested to make this choice (of technological tools) is to use more general frameworks with more comprehensive foundations for making the evaluation of educational software [46].

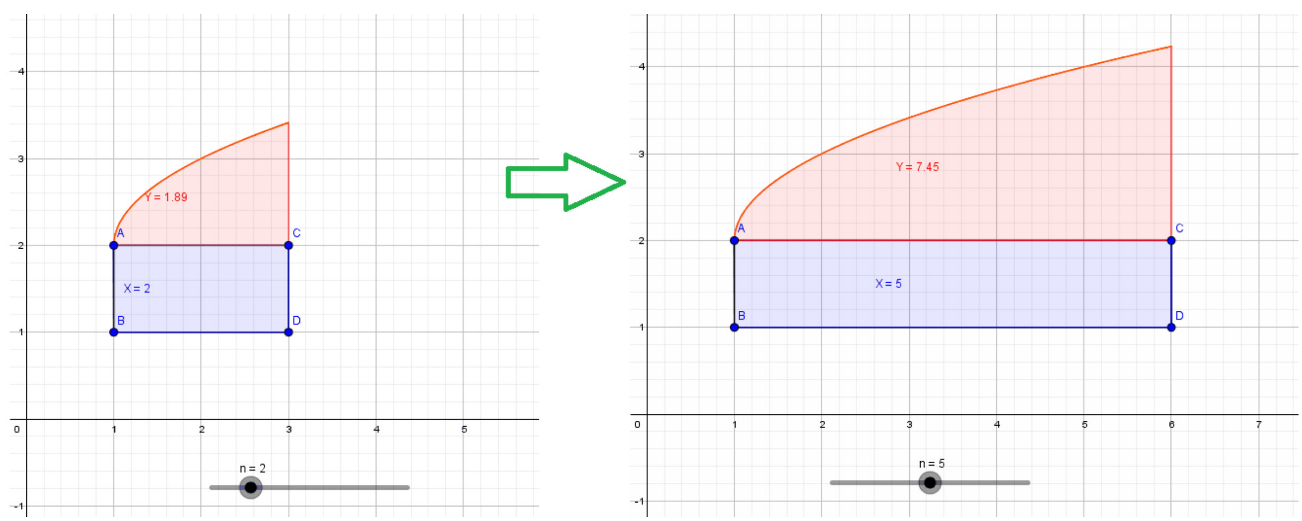
Then to fulfill these goals, GeoGebra emerges as a good alternative. This is a free software, created in Austria by Markus Hohenwarter between 2001 and 2002 as part of his master's thesis in mathematics education and computer science at the University of Salzburg, which can be used on computers with Windows, Mac OS and Linux, and also has applications for mobile phones on Android, iPad and Windows [47]. Some of the advantages of this software are: GeoGebra is an operational program on various platforms, which is available in Spanish, it's freely accessible, it has a rich database, with varied examples available on the internet, and which shows in parallel the algebraic and geometric part helping the student to see the connection between equations and graphs [48] [49] [50], without forgetting that an attractive multimedia tool allows to capture the students' attention [51]. However, the most outstanding thing about GeoGebra is that it allows you to make dynamic illustrations. Indeed, with its slider option you can show functional graphs that are not static, and how the variables change, how the area under a curve increases, and how the tangent line changes on the curve as the point varies on the horizontal axis, etc. Therefore, GeoGebra is a technological tool that allows improving the visualization in two or three dimensions regardless of the education student level [47], [48]. Remember the main goal of visualization: to allow the student to make their own conclusions, deepening their theoretical knowledge [48].

Second, the versions of Newton's FTC were presented in this writing from his manuscript "The October 1666 Tract on Fluxions" [3], [23], [52] which corresponds to the dynamic part of the theorem, and Leibniz's version of the year 1693 published in "Acta Eruditorum" [4], [12], [21], [25] which corresponds to the evaluative part. The question that immediately emerges is: how can this make a transformation in the teaching of FTC?

One answer could be to offer an alternative way of explaining it at the university setting, joining the historical presentations of Newton and Leibniz, maintaining their geometrical and dynamical points of view, with a better accessibility of this mathematical object to engineering students, via the use of technological tools.

Also, using arguments from the 17th century with mathematical software allows teachers to take advantage of this source of means offered by historical development, and helping them to improve the understanding of the mathematical object in question. In Figure 7, it is shown how using GeoGebra the rate of change of the area under a curve can be taught, preserving the argument of using quadratures to find the area under a curve.

Figure 7. How GeoGebra can be favored the visualization of Newton's FTC.



Source: own.

Although it is true that the three aforementioned relations should to be investigated deeper, it is also true that engineer trainers can take advantage of the elements that they offer: awareness that there are several ways of presenting the FTC in universities textbooks; there are different historical entries to introduce it, for instance, Barrow or Archimedes, Ibn-Qurra and Fermat's works [15]; and offer the chance to discuss which software offers better teaching possibilities to their students, etc [53] [54].

5. Conclusions

The three relations presented above can be investigated independently or by looking for a way to articulate them.

The first relation (the different ways of presenting the FTC from the college texts) of this paper suggests to the teachers that they should have a critical sense of the proposal raised in each university textbook, they should analyze which of them agrees with the curriculum of the institution where they work and what competencies are intended to be strengthened.

Studying the second relation (the historical one) it is evident that there are several alternatives to approach a mathematical object: which one based on the historical papers, or base in an author, or in a specific period of time, etc. Then the teacher must take a position on what they want to teach and how they want to teach it.

The choice and use of educational software in the classroom have repercussions on teaching, because every software was created for reaching a specific object. Then, teachers should be making a deep reflection for choosing a software in order to get the best into the classroom.

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