UNIVERSIDAD DISTRITAL
FRANCISCO JOSÉ DE CALDAS

Visión Electrónica

Más que un estado sólido

https://doi.org/10.14483/issn.2248-4728

VISIÓN ELECTRONICA
A context vision

Historical development of subseries of the harmonic series

Recorrido histórico de las subseries de la serie armónica

Edilberto Sarmiento-Sarmiento [1], Carmen Leonor Pulido-Segura [2], Andrés Jhovany Riaño-Pulido [3]

INFORMACIÓN DEL ARTICULO

Enviado: 05/12/2019
Recibido:18/12/2020
Aceptado: 04/01/2019

Keywords:
Harmonic series
Kempner series
Subsums of harmonic serie

Palabras clave:
Serie armónica
Series de kempner
Subsumas de la serie armónica

ABSTRACT:

A historical development of the harmonic series subseries that are convergent is made. It is well known that this series diverges, the series of Kempner (1914) and Irwin (1916) which are obtained by removing from the harmonic series a certain amount of numbers containing the digit 9, since that, several authors have analyzed the variations of this idea, determining the convergence of similar sub-sums of the harmonic series and calculating or estimating the sums when they are convergent, until Lubeck and Ponomarenko (2018) obtain a result that characterizes the converging subsums of the harmonic series.

RESUMEN

Se hace un recorrido histórico de las subseries de la serie armónica que son convergentes, ya que es bien conocido que está serie diverge, se presentan las series de Kempner (1914) e Irwin (1916) que se obtienen eliminando de la serie armónica una cierta cantidad de números naturales que contengan el digito 9, desde entonces, varios autores han analizado las variaciones de esta idea, determinando la convergencia de subsumas similares de las series armónicas y calculando o estimando las sumas cuando son convergentes, hasta que Lubeck y Ponomarenko (2018) obtienen un resultado que caracteriza las subseries convergentes de la serie armónica

1 MSc. in Mathematics, Universidad Nacional de Colombia, Colombia. Current position: Professor at Universidad Distrital Francisco José de Caldas, Colombia. E-mail: esarmiento@udistrital.edu.co ORCID: https://orcid.org/0000-0002-2585-5983
2 MSc. in Mathematics, Universidad Nacional de Colombia, Colombia. Current position: Professor at Universidad Distrital Francisco José de Caldas, Colombia. E-mail: clpulidos@udistrital.edu.co ORCID: https://orcid.org/0000-0002-6893-8930
3 MSc. in Mathematics, Universidad Nacional de Colombia, Colombia. Current position: Professor at Universidad Distrital Francisco José de Caldas, Colombia. E-mail: ajrianop@unal.edu.co ORCID: https://orcid.org/0000-0002-2641-8750
Cite this article as: E. Sarmiento-Sarmiento, C. L. Pulido-Segura and A. J. Riaño-Pulido, "Historical development of subseries of the harmonic series", Visión

1. Introduction

We do an historical review of subseries of the harmonic series, beginning with Kempner paper in 1914 [6], where the author made a proof of the series obtained avoiding every term which contain the digit nine in the harmonic series, this new series has a different behaviour contrary to the tendency of the harmonic series, because it results a convergent series, even better its sum is less than 90. Two years later Irwing in 1916 [5] generalized the Kempner's results and prove the convergence of subseries obtained from harmonic series, erasing any other digit a fixed number of times and bound Kempner's series in a more precise way. Several years after, Farhi in [4] studied the convergence and the sum of series where its denominators contains exactly n times a digit d. Then Baille in [1] and [8] generalized the Kempner series after vanishing of the harmonic series terms whose contain any fixed chain of digits and he obtained algorithms which allows him calculate accurately the sum of this kind of subseries of the harmonic seires. Finally, Ponomarenko in 2018 [7] found a result which has characterized the convergence of the convergence of the subseries of the harmonic series.

2. Development of the topic

The harmonic series is the sum of the multiplicative inverse of the positive integer numbers,

$$\sum_{n=1}^{\infty}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\cdots$$

This name is derived from the concept of harmonics in music: the wavelengths of the harmonics of a vibrating wire are $\frac{1}{2},\frac{1}{3},\frac{1}{4}$, etc., the fundamental wavelength of the wire.

Each term of the series after the first term is the harmonic measure of the neighbouring terms. The divergence of the harmonic series was firstly proven in the XIV century by Nicole Oresme. Other proofs were given in the XVII century by Pietro Mengoli, Johann Bernoulli and Jacob Bernoulli [2], [3].

Historically, the harmonic series has been subject to much popularity between architects, particularly in the Baroque age, when it was a tool to establish some proportions of the floor plant, elevations and to make some harmonic relations among the inside and outside architectonical features of churches and palaces.

In 1737 Euler proved that the series obtained by the harmonic series erasing some multiplicative inverse of positive integer numbers and only taking the multiplicative inverse of prime numbers which is divergent [2].

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{17}+\frac{1}{19}+\frac{1}{23}+\cdots$$

There exist some series obtained from the harmonic series erasing some denominators which are convergent, for example

$$\sum_{k=1}^{\infty}\frac{1}{2^k}=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}+\cdots=1$$

for each b>1 the geometric series whose integer ratio is $\frac{1}{b}$, are convergent

$$\sum_{k=1}^{\infty}\frac{1}{2^k}=\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}+\frac{1}{2^5}+\frac{1}{2^6}++\cdots=1$$

2.1. Kempner series

The Kempner series is a modification of the harmonic series, which vanishes every terms whose denominators written in 10 basis contains at least the digit 9, i.e. the series

$$\sum_{n\in N_9}\frac{1}{n}=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\cdots+\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{18}+\frac{1}{20}+\cdots+\frac{1}{28}+\frac{1}{30}+\cdots+\frac{1}{88}+\frac{1}{100}+\cdots$$

where $n\in N_9$ it indicates that n only takes integers positive values whose expression in the decimal basis doesn't contain any 9. This series was studied by A.J. Kempner in 1914 [6].

This series is interesting, because contrary to the harmonic series and against to the intuition (due to apparently it is erasing a little bit of denominators of positive integer numbers), it is a convergent series, even better Kempner proved that the sum is less than 90.

2.2. Kempner's Proof

$$\sum_{n\in N_9}\frac{1}{n}=\underbrace{\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)}_{a_1}+\underbrace{\left(\frac{1}{10}+\frac{1}{11}+\cdots+\frac{1}{88}\right)}_{a_2}+\underbrace{\left(\frac{1}{100}+\frac{1}{101}+\cdots+\frac{1}{888}\right)}_{a_3}+\cdots$$

Note that the the biggest fraction in$a_n$is the first term and that$\frac{1}{10^{n-1}}$contains less than terms, so that $a_n<\frac{9^n}{10^{n-1}}$

$$\sum_{n\in N_9}\frac{1}{n}=\sum_{n=1}^{\infty}a_n<\sum_{n=1}^{\infty}\frac{9^n}{10^{n-1}}=90$$

Irwin in 1916 [5] proved that erasing terms in the harmonic series whose denominators contains the digit 9 a times and at the same time the digit 8 at least b times and so successively, the digit 0 at least j times (where a,b,c,d,e,f,j,h,i,j are given greater integer numbers or equals to zero), so this series is convergent, in order to make the proof, several combinatorial arguments were used. Also, he improves the bounded of the Kempner series sum, due to was proved that

$$22.4\leq\sum_{n\in N_9}\frac{1}{n}\leq 23.3$$

A. D.Wadhwa in 1978 [9] considers the sequence $s_k=\Sigma\frac{1}{n}$ where n has exactly k zeros. He shown that s_k is a strictly decreasing sequence and that $s_k>19{,}28$, for all $k\geq 0$.
Baillie in 1978 [1] obtains a result of the Kempner series with a precision of twenty decimals. The result of the convergence of the series is 22.92067 66192 64150 34816, which is the numbered sequence A082838 in the webpage The on-line encyclopedia of integer sequences OEIS.
Schmelzer and Baillie [8] proved that the subseries of the harmonic series obtained vanishing the denominators which contains a fixed chain X of digits, converge and also they built a more efficient algorithm in order to obtain the calculus of the sum of this series. For instance, the sum of $\Sigma_{n=1}^{\infty}\frac{1}{n}$ for all n which don't contain the chain "42" in its decimal writing is 228.44630 41592 30813 25415. Other example, a Little bit complicate it is found when we calculate the sum for all n which don't contain the chain "314159" is 2302582.33386 37826 07892 02376 (Every numeric values given before are rounded to its last decimal digit).
We can use exactly the same argument with any other omitted digit and the result remains true if some summands are vanished which contains the chain of k digits in its decimal expression. For example, in the case in which we omitted every term whose denominators contains the chain "42". This result can be proven almost in the same way. Firstly, we take into account that working with numbers on base10 instead of the usual base which is 10. In this new base, each set of k digits of the expression on base 10 represent an only digit on the base 10^k, for this reason the chain of characters to substract is given by an unique "digit" in the base used. Adapting the proof given before in the base 10 to the base 10^k it is proved that this series also converge. Getting back to the base 10 we see that this series contains all the denominators which are omitted for any given chain of characters, just as denominators which include such chain if this is not represented by a "k-digit" in a base 10^k For example, if we vanish the terms with a "42", in base 100 it is possible to omit the terms 4217 and 1742, but cannot omit the term 7421. Therefore, the value of this series always is more than the series in which we omit all the denominators with a "42".
Farhi [4] studies the generalized Kempner series. In particular, he studies the series whose denominators of positive integers which exactly has n times a digit, for example, the series whose denominators exactly contain a number nine:

$$\frac{1}{9}+\frac{1}{19}+\frac{1}{29}+\frac{1}{39}+\frac{1}{49}+\frac{1}{59}+\frac{1}{69}+\frac{1}{79}+\frac{1}{89}+\cdots+\cdots$$

$$+\frac{1}{91}+\frac{1}{92}+\cdots+\frac{1}{98}+\frac{1}{109}+\frac{1}{119}+\cdots+\frac{1}{189}$$

He consider the generalized Kempner series, as the sums $s\ (d,\ n)$ of the denominators of positive integers which contain exactly n times the digit d where $0\geq d\geq 9$ (such a way the original Kempner series is $s\ (9{,}0)$) He proved for each d, the family of sequences $s\ (d{,}n)$, for $n\geq 1$ is decreasing and converge to $10\ln 10$. It is interesting that the sequence is no decreasing when it starts in $n=0$; for instance, for instance, for the original Kempner series we have that:

$$S(9{,}0)\approx 22.921<23.026\approx 10\ln 10<S(9,n)\ para\ n\geq 1$$

The Kempner series $S(9{,}0)$ converge slowly. Baillie [1] proved that, if it is summed 10^{27} terms, the error is even greater than 1.
In 1978 Baillie [1] published an efficient method to calculate the ten sums which are built when it is erased those terms which contain the digits d, with $0\geq d\geq 9$, The sum in the case d=9 is approximately 22.92067. But the sum of every terms with denominators 10^{27} still differs from the last sum more than 1.
Then in 2008 Baillie [8] developed a recursive method which allows to write the contribution of each block of k+1 digits as a function of the distributions of block of k digits for any choice of omitted digits. So that the calculus is even faster and it is used to compute sums whose denominators vanish chains of two or more digits and also sums of $\frac{1}{s}$ where s cannot contain odd digits, even digits, chains as "42" o "314159" or even combinations of such restrictions.
The convergence of this subseries were analised by Ponomarenko in 2018 [7], where he found a characterizations of the convergent subseries of the harmonic series. So that he defined that a subset A from N is r-convergent if the series $\Sigma_{n\in A}\frac{1}{n}$ converges.
Let x be in $\mathbb{R}$ define $A(x)=|\{a\in A: a\leq x\}|$ the cardinal of x. A measure commonly used of A is the asintotic density, which is defined as.

$$d(A)=\lim_{x\to\infty}\frac{A(x)}{x}$$

The mean results is that, if $d(A)=0$, then

$$\sum_{a_k\geq 1}\frac{1}{a_k}=\int_1^{\infty}\frac{A(x)}{x}dx$$

In particular, A is r-convergent if and only if $\int_1^{\infty}\frac{A(x)}{x}dx$ converges. He also generalized the results according to the convergence of Kempner series as follow: If $\lambda\in[0{,}1]$,it is denote

$$A^{\lambda}=\{n\in N:(\#9\ en\ n)\leq\lambda(\#\ digits\ in\ n)\}$$

The special case A^0 is included and has a correspondence with the original Kempner series, A^1 is the harmonic series.

In addition, he proved that A^{λ} r-convergent is and only if $\lambda < \frac{1}{10}$ and the result generalizes it for any other digit d instead of 9.

3. Conclusions

Although the harmonic series diverges, it is impressive that there exist a lot of subseries of the harmonic series such that they are convergent when we vanish some terms with some particular features, for instance, the Kempner series, which erases all the elements of the series that contains at least a nine as a digit.

It is amazing how a particular topic generates a big amount of papers for mathematicians in 100 years

It is interesting how to see that every papers starts to generalize previous results; in such a way the topic has a historical development through the Kempner ideas in 1914 which has motivated until 2018 the characterization of convergent subseries from the harmonic series.

On each paper, there are exposed some open problems which are related to the calculus of this kind of sums and with the implementation of successful algorithms to compute the sum of subseries.

References

[1] R. Baillie, “Sums of reciprocals of integers missing a given digit”, *The American Mathematical Monthly*, vol. 86, no. 5, pp. 372-374, 1979. https://doi.org/10.1080/00029890.1979.11994810

[2] G. H. Behforooz, “Thinning out the harmonic series”, *Mathematics Magazine*, vol. 68, no. 4, pp. 289-293, 1995. https://doi.org/10.1080/0025570X.1995.11996341

[3] C. H. Edwards, “The Historical Development of the Calculus”, New York: Springer-Verlag, 1979.

[4] B. Farhi, “A curious result related to Kempner's series”, *The American Mathematical Monthly*, vol. 15, no. 10, pp. 933-938, 2008. https://doi.org/10.1080/00029890.2008.11920611

[5] F. Irwin, “A curious convergent series”, The American Mathematical Monthly, vol. 23, no. 5, pp. 149-152, 1916. https://doi.org/10.1080/00029890.1916.11998196

[6] A. J. Kempner, “A curious convergent series”, The American Mathematical Monthly, vol. 21, no. 2, pp. 48-50, 1914. https://doi.org/10.1080/00029890.1914.11998004

[7] B. Lubeck and V. Ponomarenko, “Subsums of the Harmonic Series”, The American Mathematical Monthly, vol. 125, no. 4, pp. 351-355, 2018. https://doi.org/10.1080/00029890.2018.1420996

[8] T. Schmelzer and R. Baillie, “Summing a curious, slowly convergent series”, The American Mathematical Monthly, vol. 115, no. 6, pp. 525-540, 2008. https://doi.org/10.1080/00029890.2008.11920559

[9] A. D. Wadhwa, “Some convergent subseries of the harmonic series”, The American Mathematical Monthly, vol. 85, no. 8, pp. 661-663, 1978. https://doi.org/10.1080/00029890.1978.11994668